Bearing-Only Formation Tracking Control of Multiagent Systems

Shiyu Zhao, Zhenhong Li, and Zhengtao Ding

***Abstract*—This paper studies the problem of bearing-only formation control of multiagent systems, where the control of each agent merely relies on the relative bearings to its neighbors. Although this problem has received increasing research attention recently, it is still unsolved to a large extent due to its highly nonlinear dynamics. In particular, the existing control approaches are only able to solve the simplest scenario where the target formation is stationary and each agent is modeled as a single integrator. The main contribution of this paper is to propose new bearing-only formation control laws to 1) track moving target formations and 2) handle a variety of agent models including single-integrator, double-integrator, and unicycle models. These control laws are an important step towards the application of bearing-only formation control in practical tasks. Both numerical simulation and real experimental results are presented to verify the effectiveness of the theoretical results.**

***Index Terms*—Bearing measurements, bearing rigidity, formation control, multiagent systems.**

I. Introduction

THIS paper studies multiagent distributed formation control that aims to steer a group of agents to form a desired geometric pattern in a distributed manner. We particularly focus on the problem where each agent is only able to measure the relative bearings to their nearest neighboring agents while relative distance or position information is unavailable. Compared to the formation control approaches that rely on relative position measurements [1], the bearing-only formation control approach poses minimal requirements on the sensing ability of each agent, and hence provides a practical solution to achieve onboard-sensor-based formation control. In practice, bearing measurements can be obtained by, for example, vision sensors [2] or wireless sensor arrays [3], [4].

Despite the recent advances on bearing-only formation control, many important problems in this area are still unsolved due to the highly nonlinear formation dynamics. In particular, the existing bearing-only control laws are merely applicable to solve the simplest scenario where the target formation is stationary and each agent is modeled as a single integrator [5]–[16]. From the practical point of view, it is necessary to study more realistic models such as double integrators or unicycles. For example, multicopter drones and ground vehicles may be approximated by double-integrator and unicycle models. Formation control laws designed for these models can generate more feasible trajectories to be tracked by real vehicles.

Another important practical problem is how to track moving target formations using bearing measurements. The ability of a formation to move is important to achieve desired navigation tasks and respond to dynamic environments to, for example, avoid obstacles. However, it is still an open problem whether bearing measurements could be used to achieve moving formations. The existing bearing-only formation control laws are designed for stationary target formations. When applied to tracking moving target formations, these control laws would result in tracking errors. More importantly, the tracking errors may diverge to *infinity*, because these control laws merely relies on bearing errors that are always bounded even when the position errors are unbounded. As a result, there is a natural saturation constraint in these bearing-only formation control laws. Unless the control gains are sufficiently large or the leader agents move sufficiently slow, the position-tracking error will diverge to infinity. Due to this problem, it is necessary to design new bearing-only control laws to track moving target formations.

In order to handle more realistic agent models and moving target formations, we started our research by revisiting a bearing-only formation control law, which is a particular form of a more general family of controllers proposed in [15]. This control law is only applicable to the single-integrator agent model and stationary target formations. However, unlike many other existing bearing-only formation control laws, it is a gradient-descent control law, which is favorable from the stability analysis point of view. This control law has not attracted sufficient attention up to now probably because its stability analysis is based on optimization techniques and challenging to generalize to cases that are more complicated. The first contribution of our study is to present a new stability analysis of this formation control law using standard Lyapunov approaches. Such a new stability analysis is nontrivial since it requires new techniques developed based on recent work of bearing localizability [17]. Our analysis reveals some new properties of the control law, such as exponential convergence rate and, more importantly, it

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lays a foundation for the design of new bearing-only control laws.

The main contribution of this paper is to propose novel bearing-only formation control laws that can handle moving target formations and a variety of agent models including single integrators, double integrators, and unicycles. In particular, for single-integrator agent models, we propose a proportional-integral control law to track moving target formations using bearing measurements. For double-integrator agent models, a new bearing-only formation control law that requires bearing measurements and the varying rates of bearings is proposed. Similar to bearing measurements, the varying rates of relative bearings can be conveniently measured by visual sensing or sensor arrays in practice. Finally, bearing-only formation control laws are proposed for unicycle agents subject to velocity saturation and other motion constraints such as obstacle avoidance. These control laws are a key step towards the application of bearing-only formation control in practical tasks. Both numerical simulation and real experimental results are presented to verify the effectiveness of the theoretical results.

The rest of the paper is organized as follows. Section II presents the problem setup and some necessary preliminary results. Sections III, IV, and V address bearing-only formation tracking control of single-integrator, double-integrator, and unicycle agent models, respectively. Simulation and experimental results are given in Section VI. Conclusions are drawn in Section VII.

II. Preliminaries and Problem Statement

A. Notations for Formation

Consider n mobile agents in $\mathbb{R}^d$ ($n \geq 2$ and $d \geq 2$). Let $p_i(t) \in \mathbb{R}^d$ be the position of agent $i \in \{1, \ldots, n\}$ at time t, and $p = [p_1^{\mathrm{T}}, \ldots, p_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$ be the configuration of the agents. The interaction among the agents is described by a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which consists of a vertex set $\mathcal{V} = \{1, \ldots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The edge $(i, j) \in \mathcal{E}$ indicates that agent i can measure the relative bearing of agent j, and hence agent j is a neighbor of i. The set of neighbors of agent i is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. This paper only considers undirected graphs where $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. A formation, denoted as $(\mathcal{G}, p)$, is $\mathcal{G}$ with its vertex i mapped to p_i for all $i \in \mathcal{V}$.

Define the *edge vector* and *bearing vector* for edge (i, j), respectively, as

$$e_{ij} := p_j - p_i, \quad g_{ij} := \frac{e_{ij}}{\|e_{ij}\|}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector or the spectral norm of a matrix. The unit vector g_{ij} represents the relative bearing of p_j with respect to p_i. Note that $e_{ij} = -e_{ji}$ and $g_{ij} = -g_{ji}$. For the bearing vector g_{ij}, define

$$P_{g_{ij}} := I_d - g_{ij} g_{ij}^{\mathrm{T}} \in \mathbb{R}^{d \times d}$$

where $I_d \in \mathbb{R}^{d \times d}$ is the identity matrix. Note that $P_{g_{ij}}$ is an orthogonal projection matrix that can geometrically project a vector onto the orthogonal compliment of g_{ij}. It can be verified that $P_{g_{ij}}$ is positive semidefinite and $\mathrm{Null}(P_{g_{ij}}) = \mathrm{span}\{g_{ij}\}$. As a result, a vector $x \in \mathbb{R}^d$ is parallel to g_{ij} if and only if $P_{g_{ij}} x = 0$. This orthogonal projection matrix is widely used in bearing-based control and estimation problems [10], [17]. Moreover, it follows from $g_{ij} = e_{ij}/\|e_{ij}\|$ that the time derivative of g_{ij} is

$$\dot{g}_{ij} = \frac{P_{g_{ij}}}{\|e_{ij}\|} \dot{e}_{ij}. \tag{1}$$

Since $P_{g_{ij}} g_{ij} = 0$, $\dot{g}_{ij}$ is orthogonal to g_{ij} and e_{ij}. As a result, $g_{ij}^{\mathrm{T}} \dot{g}_{ij} = 0$ and $e_{ij}^{\mathrm{T}} \dot{g}_{ij} = 0$.

Oriented graphs are widely used in this paper. Specifically, an *orientation* of an undirected graph is the assignment of a direction to each edge. An *oriented graph* is an undirected graph together with an orientation [18]. Consider an arbitrary oriented graph of $\mathcal{G}$. Let m be the number of undirected edges in $\mathcal{G}$. Hence, the oriented graph has m directed edges. Suppose edge (i, j) in $\mathcal{G}$ corresponds to the kth directed edge in the oriented graph where $k \in \{1, \ldots, m\}$. The edge and bearing vectors for the kth directed edge can be expressed as

$$e_k := e_{ij} = p_j - p_i, \quad g_k := \frac{e_k}{\|e_k\|}.$$

Similarly, $\dot{g}_k = P_{g_k} \dot{e}_k / \|e_k\|$, $g_k^{\mathrm{T}} \dot{g}_k = 0$, and $e_k^{\mathrm{T}} \dot{g}_k = 0$. Denote $e = [e_1^{\mathrm{T}}, \ldots, e_m^{\mathrm{T}}]^{\mathrm{T}}$ and $g = [g_1^{\mathrm{T}}, \ldots, g_m^{\mathrm{T}}]^{\mathrm{T}}$. Let $H \in \mathbb{R}^{m \times n}$ be the *incidence matrix* of the oriented graph. Specifically, in the kth row of H, $[H]_{ki} = -1$ since vertex i is the tail of edge k, $[H]_{kj} = 1$ since vertex j is the head of edge k, and all the other entries in the kth row are zero. By definition

$$e = (H \otimes I_d) p := \bar{H} p$$

where $\otimes$ denotes the Kronecker product. For a connected graph, it holds that $H \mathbf{1}_n = 0$ and $\mathrm{rank}(H) = n - 1$ [18], where $\mathbf{1}_n = [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$.

Without loss of generality, suppose the first n_ℓ agents are leaders and the rest $n_f = n - n_\ell$ agents are followers. Let $\mathcal{V}_\ell = \{1, \ldots, n_\ell\}$ and $\mathcal{V}_f = \mathcal{V} \setminus \mathcal{V}_\ell$ be the sets of leaders and followers, respectively. The positions of the leaders and followers are denoted as $p_\ell = [p_1^{\mathrm{T}}, \ldots, p_{n_\ell}^{\mathrm{T}}]^{\mathrm{T}}$ and $p_f = [p_{n_\ell+1}^{\mathrm{T}}, \ldots, p_n^{\mathrm{T}}]^{\mathrm{T}}$, respectively. Then $p = [p_\ell^{\mathrm{T}}, p_f^{\mathrm{T}}]^{\mathrm{T}}$. The velocities of the leaders and followers are denoted as $v_\ell = \dot{p}_\ell$ and $v_f = \dot{p}_f$, respectively.

B. Target Formation

The desired target formation that the agents should achieve is described as below.

Definition 1 (Target formation): The target formation $(\mathcal{G}, p^*(t))$ satisfies the constant interneighbor bearings $\{g_{ij}^*\}_{(i,j) \in \mathcal{E}}$ and the time-varying leader positions $\{p_i^*(t)\}_{i \in \mathcal{V}_\ell}$. The leaders move at a common constant velocity $v_c \in \mathbb{R}^d$.

The target formation in Definition 1 is jointly determined by the constant bearings and moving leaders. It has two key properties. The first property is existence. In this paper, we only consider feasible bearings and leader positions so that the target formation defined above exists. Feasible bearings and leader positions can be easily obtained from a formation that has the desired geometric pattern.

The second key property of the target formation is uniqueness. In order to ensure the uniqueness of the target formation, the

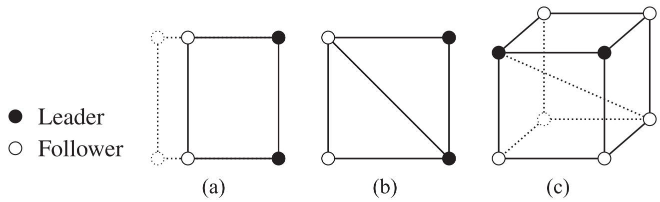

Fig. 1. Target formation in (a) cannot be uniquely determined by the bearings and leader positions. The target formations in (b) and (c) are unique. The target formation in (c) is a 3-D cube.

bearings and leader positions must satisfy certain conditions. To characterize these conditions, we need to introduce an important matrix termed *bearing Laplacian* [17]. For the target formation, define a matrix $\mathcal{B} \in \mathbb{R}^{dn \times dn}$ with the ijth block of submatrix as

$$[\mathcal{B}]_{ij} = \begin{cases} \mathbf{0}_{\mathbf{d}\times\mathbf{d}}, & i \neq j,\ (i,j) \notin \mathcal{E}, \\ -P_{g^*_{ij}}, & i \neq j,\ (i,j) \in \mathcal{E}, \\ \sum_{k\in\mathcal{N}_i} P_{g^*_{ik}}, & i = j,\ i \in \mathcal{V}. \end{cases}$$

The bearing Laplacian matrix $\mathcal{B}$ is a matrix-weighted Laplacian that characterizes both the underlying graph and the bearings of the target formation. For undirected graphs, it holds that $\mathcal{B}$ is symmetric positive semidefinite and $\mathrm{span}\{p^*, \mathbf{1}_n \otimes I_d\} \subseteq \mathrm{Null}(\mathcal{B})$ [17]. According to the partition of leader and follower agents, we can partition $\mathcal{B}$ to

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{\ell\ell} & \mathcal{B}_{\ell f} \\ \mathcal{B}_{f\ell} & \mathcal{B}_{ff} \end{bmatrix}$$

where $\mathcal{B}_{ff} \in \mathbb{R}^{dn_f \times dn_f}$. The uniqueness of the target formation could be characterized by the bearing Laplacian as shown in the following lemma.

Lemma 1 (Condition for unique target formation [17]): The target configuration $p^*(t)$ can be uniquely determined by the bearings $\{g^*_{ij}\}_{(i,j)\in\mathcal{E}}$ and leader positions $\{p^*_i\}_{i\in\mathcal{V}_\ell}$ if and only if $\mathcal{B}_{ff}$ is nonsingular.

When $\mathcal{B}_{ff}$ is nonsingular, the position and velocity of the followers in the target formation are uniquely determined as $p^*_f(t) = -\mathcal{B}^{-1}_{ff}\mathcal{B}_{f\ell}p^*_\ell(t)$ and $v^*_f(t) = -\mathcal{B}^{-1}_{ff}\mathcal{B}_{f\ell}v^*_\ell(t)$ [17]. In this paper, the leaders have the same common velocity v_c, i.e., $v^*_\ell(t) = \mathbf{1}_{n_\ell} \otimes v_c$. It can be calculated that $v^*_f(t) = \mathbf{1}_{n_f} \otimes v_c$ so that all the followers in the target formation move at the same common velocity [19, Th. 2]. Let $\bar{p}^*(t) := \sum_{i=1}^n p^*_i(t)/n$ and $\tilde{p}^* := p^*(t) - \mathbf{1}_n \otimes \bar{p}^*(t)$. When the target formation is unique, the vector $\tilde{p}^*$ is constant and represents the unique geometric pattern of the formation. The vector $\bar{p}^*(t)$ is time-varying and represents the centroid of the target formation that moves at the constant velocity v_c.

Illustrative examples of nonunique and unique formations are shown in Fig. 1. In order to ensure a unique target formation, there must exist sufficient and appropriate leaders. It is worth noting that at least two leaders are required to ensure a unique target formation. Details on leader selection can be found in [17]. More examples and other conditions for uniqueness can be found in [17, Sec. 4]. Note that uniqueness is called bearing localizability in the context of network localization [17].

This paper only considers unique target formations. Otherwise, the target formation is not guaranteed to achieve by any control approaches. Hence, we make the following assumption.

Assumption 1 (Unique target formation): Assume that the target formation $(\mathcal{G}, p^*)$ in Definition 1 is unique, i.e., $\mathcal{B}_{ff}$ is positive definite.

C. Problem Statement

The control problem to be solved in this paper is to steer the agents to achieve the target formation based merely on bearing measurements. This problem is formally stated as follows.

Problem 1 (Bearing-only formation tracking control): Design control input for agent $i \in \mathcal{V}_f$ based merely on the bearing measurements $\{g_{ij}(t)\}_{j\in\mathcal{N}_i}$ and the varying rate of the bearings $\{\dot{g}_{ij}(t)\}_{j\in\mathcal{N}_i}$ such that $g_{ij} \to g^*_{ij}$ for all $(i,j) \in \mathcal{E}$ as $t \to \infty$.

Several remarks on Problem 1 are given below. First, since the target formation is uniquely determined by the bearings (and leader positions), once the bearings reach the desired values, the target formation is achieved. Second, it is only required to control the followers whereas the leaders are assumed to be controlled properly so that they have desired positions. The reason why we do not consider the coordination of the leaders is that there are merely a very small number of leaders such as two, which is different from containment control problems where there may exist many leaders which require sophisticated distributed coordination [20]. Third, although the underlying graph of the entire formation is assumed to be undirected, leaders are controlled independently and do not use the information of their neighbors whereas the control of a follower does rely on the information of its neighbors.

Interagent collision avoidance is an important problem in multiagent formation control. This problem, however, has not been considered in many existing formation control approaches [1], [21], because the system convergence would become extremely difficult to analyze if it is considered. In fact, given a specific formation control law, interagent collision is determined by the initial formation condition. However, it is nontrivial to identify those initial conditions that would lead to interagent collision even for simple linear formation dynamics. As a result, many existing formation control approaches rely on an *implicit* assumption of interagent collision avoidance. In this paper, we also adopt the collision avoidance assumption.

Assumption 2 (Interagent collision avoidance): Assume no neighboring agents collide with each other during the formation evolvement.

Assumption 2 ensures that the bearing vector between any pair of neighbors is always well defined during formation evolvement. Without this assumption, the convergence analysis of the control laws proposed in the rest of the paper is still valid, but only before collision occurs. In this paper, we also present sufficient conditions to simultaneously guarantee collision avoidance and system convergence. With these conditions, this collision-free assumption could be dropped.

In the rest of the paper, three types of agent dynamical models will be studied: single-integrator, double-integrator, and unicycle.

III. Bearing-Only Formation Tracking Control: Single-Integrator Agents

This section studies single-integrator agent models: $\dot{p}_i(t) = u_i(t)$, where $u_i(t)$ is the velocity input to be designed. The first subsection addresses the case of stationary target formations and reanalyzes a control law proposed in [15]. The second subsection generalizes this control law to track moving target formations.

A. Stationary Target Formation

Suppose the leaders are stationary (i.e., $v_c = 0$). For the leaders, it holds that $\dot{p}_i(t) = 0$ for $i \in \mathcal{V}_\ell$. For the followers, consider the bearing-only control law

$$\dot{p}_i(t) = \sum_{j \in \mathcal{N}_i} \left(g_{ij}(t) - g_{ij}^*\right), \quad i \in \mathcal{V}_f. \tag{2}$$

Control law (2) is a particular form of a general family of controllers designed for bearing-only and distance-based formation control tasks proposed in [15, eq. (13)]. Its asymptotic stability has been analyzed using optimization techniques in [15]. The novelty of this subsection is to present a new stability analysis based on Lyapunov approaches. This stability analysis reveals some new properties of the control law, such as exponential convergence rate and lays a foundation for designing new bearing-only control laws.

Consider an arbitrary oriented graph and suppose H, $g(t)$, and g^* are the corresponding incidence matrix, bearing vectors at t, and desired bearing vectors, respectively. Then, control law (2) can be rewritten in a matrix-vector form as

$$\dot{p} = -\begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*) \tag{3}$$

whose initial state is $p(0) = [(p_\ell^*)^{\mathrm{T}}, p_f^{\mathrm{T}}(0)]^{\mathrm{T}}$ where $p_f(0)$ could be arbitrary. To analyze the formation stability, we first introduce two useful lemmas about the key quantity $p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*)$.

Lemma 2: Suppose no agents coincide in p^* or p. It holds that

$$p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) \geq 0 \tag{4}$$

$$(p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) \leq 0 \tag{5}$$

$$(p - p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) \geq 0 \tag{6}$$

where the equalities hold if and only if $g = g^*$.

Proof: First,

$$\begin{aligned} p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) &= e^{\mathrm{T}}(g - g^*) \\ &= \sum_{k=1}^{m}(e_k^{\mathrm{T}} g_k - e_k^{\mathrm{T}} g_k^*) \\ &= \sum_{k=1}^{m}\|e_k\|(1 - g_k^{\mathrm{T}} g_k^*) \\ &= \frac{1}{2}\sum_{k=1}^{m}\|e_k\|\|g_k - g_k^*\|^2 \geq 0. \end{aligned}$$

Since $\|e_k\| \neq 0$, we know $p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) = 0$ if and only if $g_k = g_k^*$ for all k. Similarly, (5) holds because $(p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) = (e^*)^{\mathrm{T}}(g - g^*) = \sum_{k=1}^{m}\|e_k^*\|((g_k^*)^{\mathrm{T}} g_k - 1) \leq 0$. Since $\|e_k^*\| \neq 0$, the equality holds if and only if $g_k = g_k^*$ for all k. Inequality (6) can be obtained by combining (5) and (4). ∎

The following result establishes the equivalence between $p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*)$ and $p^{\mathrm{T}}\mathcal{B}p$.

Lemma 3: Suppose no agents coincide in p^* or p. It holds that

$$p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) \geq \frac{1}{2\max_k \|e_k\|} p^{\mathrm{T}}\mathcal{B}p \tag{7}$$

where $\mathcal{B}$ is the bearing Laplacian of the target formation $(\mathcal{G}, p^*)$. When $g - g^*$ is sufficiently small so that $g_k^{\mathrm{T}} g_k^* \geq 0$ for all k, it holds that

$$p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) \leq \frac{1}{\min_k \|e_k\|} p^{\mathrm{T}}\mathcal{B}p. \tag{8}$$

Proof: Note that $\mathcal{B}$ can be expressed as $\mathcal{B} = \bar{H}^{\mathrm{T}} D(P_{g_k^*})\bar{H}$ where $D(P_{g_k^*}) = \mathrm{blkdiag}(P_{g_1^*}, \ldots, P_{g_m^*})$ [17, Lemma 2]. It follows that:

$$\begin{aligned} p^{\mathrm{T}}\mathcal{B}p &= p^{\mathrm{T}}\bar{H}^{\mathrm{T}} D(P_{g_k^*})\bar{H}p = e^{\mathrm{T}} D(P_{g_k^*})e \\ &= \sum_{k=1}^{m} e_k^{\mathrm{T}}(I_d - g_k^*(g_k^*)^{\mathrm{T}})e_k = \sum_{k=1}^{m}\|e_k\|^2(1 - (g_k^{\mathrm{T}} g_k^*)^2) \\ &= \sum_{k=1}^{m}\|e_k\|^2(1 - g_k^{\mathrm{T}} g_k^*)(1 + g_k^{\mathrm{T}} g_k^*). \end{aligned} \tag{9}$$

Since $1 + g_k^{\mathrm{T}} g_k^* \leq 2$, it follows from (9) that:

$$\begin{aligned} p^{\mathrm{T}}\mathcal{B}p &\leq 2\max_k \|e_k\| \sum_{k=1}^{m}\|e_k\|(1 - g_k^{\mathrm{T}} g_k^*) \\ &= 2\max_k \|e_k\| p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*). \end{aligned}$$

Inequality (7) follows immediately.

Suppose that $g - g^*$ is sufficiently small so that $g_k^{\mathrm{T}} g_k^* \geq 0$ for all k (i.e., the angle between g_k and g_k^* is no greater than $\pi/2$). Since $1 + g_k^{\mathrm{T}} g_k^* \geq 1$, it is implied by (9) that

$$\begin{aligned} p^{\mathrm{T}}\mathcal{B}p &\geq \min_k \|e_k\| \sum_{k=1}^{m}\|e_k\|(1 - g_k^{\mathrm{T}} g_k^*) \\ &= \min_k \|e_k\| p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*). \end{aligned}$$

Inequality (8) follows immediately. ∎

With the above two lemmas, we prove the exponential stability of (3) as follows. The position error is defined as

$$\delta_p(t) = p(t) - p^*.$$

The objective is to prove that $\delta_p(t) \to 0$ as $t \to \infty$.

Theorem 1 (Single-integrator formation stabilization): Under Assumptions 1 and 2, $p(t)$ converges to p^* exponentially fast by the action of control law (3).

Proof: Note that $\delta_p = [0, \delta_{p_f}^{\mathrm{T}}]^{\mathrm{T}}$ since $p_\ell = p_\ell^*$. As a result,

$$\delta_p^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} = \delta_p^{\mathrm{T}}.$$

Consider the Lyapunov function

$$V = \frac{1}{2}\|\delta_p\|^2.$$

The derivative of V along the system trajectory is

$$\begin{aligned}\dot{V} &= \delta_p^{\mathrm{T}}\dot{\delta}_p = \delta_p^{\mathrm{T}}\dot{p}\\ &= -\delta_p^{\mathrm{T}}\begin{bmatrix}0 & 0\\ 0 & I_{dn_f}\end{bmatrix}\bar{H}^{\mathrm{T}}(g-g^*)\\ &= -\delta_p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\\ &= -(p-p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\\ &= -p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*) + (p^*)^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\\ &\le -p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*) \end{aligned} \tag{10}$$

where the last inequality is due to (5). Substituting (7) into (10) gives

$$\dot{V} \le -\frac{1}{2\max_k \|e_k\|}p^{\mathrm{T}}\mathcal{B}p \le 0. \tag{11}$$

Since $\mathcal{B}p^* = 0$, we have $p^{\mathrm{T}}\mathcal{B}p = (p-p^*)^{\mathrm{T}}\mathcal{B}(p-p^*) = \delta_p^{\mathrm{T}}\mathcal{B}\delta_p$. Furthermore, since $\delta_p = [0, \delta_{p_f}^{\mathrm{T}}]^{\mathrm{T}}$, we have $\delta_p^{\mathrm{T}}\mathcal{B}\delta_p = \delta_{p_f}^{\mathrm{T}}\mathcal{B}_{ff}\delta_{p_f} \ge \lambda_{\min}(\mathcal{B}_{ff})\|\delta_{p_f}\|^2 = \lambda_{\min}(\mathcal{B}_{ff})\|\delta_p\|^2$. Substituting into (11) gives

$$\dot{V} \le -\frac{\lambda_{\min}(\mathcal{B}_{ff})}{2\max_k \|e_k\|}\|\delta_p\|^2. \tag{12}$$

Note that

$$\begin{aligned}\max_k \|e_k\| \le \|e\| = \|\bar{H}p\| &= \|\bar{H}(p-p^*+p^*)\|\\ &\le \|\bar{H}\delta_p\| + \|\bar{H}p^*\|\\ &= \|\bar{H}\delta_p\| + \|\bar{H}\tilde{p}^*\|\\ &\le \|\bar{H}\|(\|\delta_p\| + \|\tilde{p}^*\|)\end{aligned} \tag{13}$$

where the last equality is due to $\bar{H}\tilde{p}^* = \bar{H}(p^* - \mathbf{1}_n \otimes \bar{p}) = \bar{H}p^*$. Since $\dot{V} \le 0$, we have $\|\delta_p(t)\| \le \|\delta_p(0)\|$. Substituting (13) into (12) yields

$$\dot{V} \le -\underbrace{\frac{\lambda_{\min}(\mathcal{B}_{ff})}{\|\bar{H}\|(\|\delta_p(0)\| + \|\tilde{p}^*\|)}}_{a}\frac{\|\delta_p\|^2}{2} := -aV \tag{14}$$

which indicates exponential convergence rate. ■

The convergence rate a in (14) is jointly determined by four parameters: 1) The graph topology described by $\bar{H}$, 2) geometric pattern of the target formation described by $\tilde{p}^*$, 3) initial error $\delta_p(0)$, and 4) the smallest eigenvalue $\lambda_{\min}(\mathcal{B}_{ff})$. Note that $\lambda_{\min}(\mathcal{B}_{ff})$ could be interpreted as a measure of the "degree of uniqueness" of the target formation, which intuitively describes how far the target formation is from being nonunique. One immediate conclusion is that the convergence rate would be large if the formation is close to the target one (i.e., $\|\delta_p(0)\|$ is small) or the degree of uniqueness is strong (i.e., $\lambda_{\min}(\mathcal{B}_{ff})$ is large). Designing optimal graph topologies and formation geometric patterns to minimize $\|\bar{H}\|$ and $\|\tilde{p}^*\|$ could also speed up convergence and it deserves further study in the future.

Note that Theorem 3 relies on the assumption of interagent collision avoidance. To remove this assumption, we next give a sufficient condition on the initial formation to simultaneously guarantee collision avoidance and formation convergence. Suppose γ is the desired minimum separation between any two agents during formation evolvement and satisfies $0 < \gamma < \min_{i,j\in\mathcal{V}} \|p_i^* - p_j^*\|$.

Corollary 1 (Sufficient condition for collision avoidance): Under Assumption 1, if the initial formation is sufficiently close to the target formation so that

$$\|\delta_p(0)\| \le \epsilon := \frac{1}{\sqrt{n}}\left(\min_{i,j\in\mathcal{V}}\|p_i^* - p_j^*\| - \gamma\right) \tag{15}$$

then $\|p_i(t) - p_j(t)\| \ge \gamma$ for all $i, j \in \mathcal{V}$ and all $t \ge 0$ and $p(t)$ converges to p^* exponentially fast.

Proof: For any $i, j \in \mathcal{V}$ and any $t \ge 0$, it holds that

$$p_i(t) - p_j(t) = [p_i(t) - p_i^*] - [p_j(t) - p_j^*] + [p_i^* - p_j^*].$$

It follows that

$$\begin{aligned}\|p_i(t) - p_j(t)\| &\ge \|p_i^* - p_j^*\| - \|p_i(t) - p_i^*\| - \|p_j(t) - p_j^*\|\\ &\ge \|p_i^* - p_j^*\| - \sum_{k=1}^{n}\|p_k(t) - p_k^*\|\\ &\ge \|p_i^* - p_j^*\| - \sqrt{n}\|p(t) - p^*\|\\ &= \|p_i^* - p_j^*\| - \sqrt{n}\|\delta_p(t)\|.\end{aligned} \tag{16}$$

At $t = 0$, substituting (15) into the above inequality gives $\|p_i(0) - p_j(0)\| \ge \gamma$ and hence there is no interagent collision in the initial formation. Since $\dot{V} \le 0$ as shown in (11) in the absence of interagent collision, $\|\delta_p(0)\| \le \epsilon$ implies $\|\delta_p(t)\| \le \epsilon$ for all t. Otherwise, there must exist an escape time t_1 such that $\|\delta_p(t_1)\| = \epsilon$ and $\dot{V}(t_1) > 0$, which is impossible. As a result, $\|p_i(t) - p_j(t)\| \ge \gamma$ is guaranteed for all t by (16). It then follows from (14) that $\|\delta_p(t)\|$ converges to zero exponentially fast. ■

B. Moving Target Formation

When the leaders move at a nonzero constant velocity v_c, we generalize (2) to propose a new control law

$$\begin{aligned}\dot{p}_i(t) = k_{\mathrm{p}}\sum_{j\in\mathcal{N}_i}\left(g_{ij}(t) - g_{ij}^*\right)\\ + k_{\mathrm{I}}\int_0^t \sum_{j\in\mathcal{N}_i}\left(g_{ij}(\tau) - g_{ij}^*\right)\mathrm{d}\tau, \quad i \in \mathcal{V}_f\end{aligned} \tag{17}$$

where k_{p} and k_{I} are constant and positive control gains. The idea of (17) is to simply introduce an integral term, yet the stability analysis is nontrivial as shown as follows.

Denote

$$\eta_i(t) = k_{\mathrm{I}}\int_0^t \sum_{j\in\mathcal{N}_i}\left(g_{ij}(\tau) - g_{ij}^*\right)\mathrm{d}\tau$$

and $\eta = [\eta_1^{\mathrm{T}}, \dots, \eta_n^{\mathrm{T}}]^{\mathrm{T}} = [\eta_\ell^{\mathrm{T}}, \eta_f^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{dn}$. Then, the matrix-vector form of control law (17) is

$$\dot{p} = -k_{\mathrm{p}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}(g - g^*) + \eta,$$

$$\dot{\eta} = -k_{\mathrm{I}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}(g - g^*).$$

The initial values satisfy

$$p(0) = \begin{bmatrix} p_\ell^*(0) \\ p_f(0) \end{bmatrix} \quad \eta(0) = \begin{bmatrix} \mathbf{1}_{n_\ell} \otimes v_c \\ \eta_f(0) \end{bmatrix}$$

where $p_f(0)$ and $\eta_f(0)$ could be arbitrary. In order to prove formation stability, define the error states as

$$\delta_p = p - p^* \quad \tilde{\eta} = \eta - \mathbf{1}_n \otimes v_c.$$

The objective is to prove that $\delta_p \to 0$ and $\tilde{\eta} \to 0$ as $t \to \infty$. To do that, it is necessary to first establish the equivalence between $e^{\mathrm{T}}(g - g^*)$ and $\|\delta_p\|$.

Corollary 2: Suppose no agents coincide in p^* or p. It holds that

$$e^{\mathrm{T}}(g - g^*) \geq \frac{\lambda_{\min}(\mathcal{B}_{ff})\|\delta_p\|^2}{2\|\bar{H}\|(\|\delta_p\| + \|\tilde{p}^*\|)}$$

$$e^{\mathrm{T}}(g - g^*) \leq 2\sqrt{m}\|\bar{H}\|(\|\delta_p\| + \|\tilde{p}^*\|). \tag{18}$$

When $g - g^*$ is sufficiently small such that $g_k^{\mathrm{T}} g_k^* \geq 0$ for all k, a tight upper bound can be obtained as $e^{\mathrm{T}}(g - g^*) \leq \lambda_{\max}(\mathcal{B}_{ff})\|\delta_p\|^2 / \min_k \|e_k\|$.

Proof: First of all, since $\delta_p = [0, \delta_{p_f}^{\mathrm{T}}]^{\mathrm{T}}$, we have $\delta_p^{\mathrm{T}} \mathcal{B} \delta_p = \delta_{p_f}^{\mathrm{T}} \mathcal{B}_{ff} \delta_{p_f}$. It follows from $\mathcal{B}p^* = 0$ that

$$p^{\mathrm{T}} \mathcal{B} p = \delta_p^{\mathrm{T}} \mathcal{B} \delta_p \geq \lambda_{\min}(\mathcal{B}_{ff})\|\delta_{p_f}\|^2 = \lambda_{\min}(\mathcal{B}_{ff})\|\delta_p\|^2 \tag{19}$$

$$p^{\mathrm{T}} \mathcal{B} p = \delta_p^{\mathrm{T}} \mathcal{B} \delta_p \leq \lambda_{\max}(\mathcal{B}_{ff})\|\delta_{p_f}\|^2 = \lambda_{\max}(\mathcal{B}_{ff})\|\delta_p\|^2. \tag{20}$$

The lower bound in (18) can be obtained by substituting (19) and (13) into (7). In order to obtain an upper bound, note that $e^{\mathrm{T}}(g - g^*) = \sum_{k=1}^m \|e_k\|(1 - g_k^{\mathrm{T}} g_k^*) \leq 2\sum_{k=1}^m \|e_k\| \leq 2\sqrt{m}\sqrt{\sum_{k=1}^m \|e_k\|^2} = 2\sqrt{m}\|e\|$, where the last inequality follows from the inequality between arithmetic and quadratic means. It follows from the upper bound of $\|e\|$ in (13) that $e^{\mathrm{T}}(g - g^*) \leq 2\sqrt{m}\|\bar{H}\|(\|\delta_p\| + \|\tilde{p}^*\|)$. This bound is not tight because it is not zero when $\|\delta_p\| = e^{\mathrm{T}}(g - g^*) = 0$.

When $g - g^*$ is sufficiently small so that $g_k^{\mathrm{T}} g_k^* \geq 0$ for all k (i.e., the angle between g_k and g_k^* is no larger than $\pi/2$), substituting (20) into (8) gives $e^{\mathrm{T}}(g - g^*) \leq \lambda_{\max}(\mathcal{B}_{ff})\|\delta_p\|^2 / \min_k \|e_k\|$. This upper bound is tight in the sense that it is zero when $\|\delta_p\| = 0$. ■

Remark 1: Inequality (18) is particularly useful in the formation stability analysis as shown later. It must be noted that this inequality is meaningful only if the target formation is unique (i.e., Assumption 1 is fulfilled). Otherwise, if the target formation is not unique and hence $\lambda_{\min}(\mathcal{B}_{ff}) = 0$, then (18) degrades to $e^{\mathrm{T}}(g - g^*) \geq 0$, which is the same as (4). In this case, δ_p could be unbounded even if $e^{\mathrm{T}}(g - g^*)$ is bounded or zero. For example, for the target formation in Fig. 1(a), the target bearings could be satisfied so that $e^{\mathrm{T}}(g - g^*) = 0$ while the left two agents can move to the left to generate arbitrarily large δ_p.

The formation stability is analyzed as follows.

Theorem 2 (Single-integrator formation tracking): Under Assumptions 1 and 2, $p(t)$ converges to $p^*(t)$ asymptotically by the action of control law (17), where $p^*(t)$ represents the target configuration moving at velocity v_c with a fixed geometric pattern as defined in Definition 1.

Proof: Since $\delta_p = [0, \delta_{p_f}^{\mathrm{T}}]^{\mathrm{T}}$ and $\tilde{\eta} = [0, \tilde{\eta}_f^{\mathrm{T}}]^{\mathrm{T}}$, we have

$$\delta_p^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} = \delta_p^{\mathrm{T}}, \quad \tilde{\eta}^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} = \tilde{\eta}^{\mathrm{T}}.$$

Consider the Lyapunov function

$$V = e^{\mathrm{T}}(g - g^*) + \frac{1}{2k_{\mathrm{I}}}\tilde{\eta}^{\mathrm{T}}\tilde{\eta} \geq 0. \tag{21}$$

Note that $V = 0$ if and only if $\tilde{\eta} = 0$ and $e^{\mathrm{T}}(g - g^*) = 0 \Leftrightarrow g = g^*$ according to (4). Since the target formation is assumed to be unique, $g = g^*$ and $p_\ell = p_\ell^*$ imply $p = p^*$. As a result, $V = 0$ if and only if $\delta_p = 0$ and $\tilde{\eta} = 0$.

Since $e^{\mathrm{T}}\dot{g} = \sum_{k=1}^m e_k^{\mathrm{T}}\dot{g}_k = 0$ by (1), the derivative of V is

$$\begin{aligned} \dot{V} &= e^{\mathrm{T}}\dot{g} + (g - g^*)^{\mathrm{T}}\bar{H}\dot{p} + \frac{1}{k_{\mathrm{I}}}\tilde{\eta}^{\mathrm{T}}\dot{\tilde{\eta}} \\ &= -k_{\mathrm{p}}(g - g^*)^{\mathrm{T}}\bar{H}\begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix}\bar{H}^{\mathrm{T}}(g - g^*) \\ &\quad + (g - g^*)^{\mathrm{T}}\bar{H}\eta - \tilde{\eta}^{\mathrm{T}}\begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix}\bar{H}^{\mathrm{T}}(g - g^*) \\ &= -k_{\mathrm{p}}(g - g^*)^{\mathrm{T}}\bar{H}\begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix}\bar{H}^{\mathrm{T}}(g - g^*) \\ &\quad + (g - g^*)^{\mathrm{T}}\bar{H}\eta - \tilde{\eta}^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g - g^*) \\ &= -k_{\mathrm{p}}(g - g^*)^{\mathrm{T}}\bar{H}\begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix}\bar{H}^{\mathrm{T}}(g - g^*) \leq 0 \end{aligned} \tag{22}$$

where the last equality is due to $\bar{H}\tilde{\eta} = \bar{H}(\eta - \mathbf{1}_n \otimes v_c) = \bar{H}\eta$. Since $\dot{V} \leq 0$, $V(t) \leq V(0)$ for all t. As a result, $e^{\mathrm{T}}(g - g^*)$ and $\|\tilde{\eta}\|$ are always bounded. By the lower bound of $e^{\mathrm{T}}(g - g^*)$ in (18), we know $\frac{\|\delta_p\|^2}{\|\delta_p\| + \|\tilde{p}^*\|}$ is also bounded from above. Suppose the upper bound is α. Then

$$\frac{\|\delta_p\|^2}{\|\delta_p\| + \|\tilde{p}^*\|} \leq \alpha. \tag{23}$$

Inequality (23) can be converted to a quadratic inequality of $\|\delta_p\|$, which further implies $\|\delta_p\| \in [0, \xi_+]$, where ξ_+ is the positive root of the corresponding quadratic equality and $\xi_+ = (\alpha + \sqrt{\alpha^2 + 4\alpha\|\tilde{p}^*\|})/2$. Hence, $\|\delta_p\|$ is always bounded. As a result, there exists a compact set of δ_p and $\tilde{\eta}$ that is invariant under the error dynamics. By the invariance principle [22, Th. 4.4], the error states converge to the set where $\dot{V}=0$. When $\dot{V}=0$, it follows from (22) that $\left[\begin{smallmatrix}0 & 0\\ 0 & I_{dn_f}\end{smallmatrix}\right]\bar{H}^{\mathrm{T}}(g-g^*)=0$, which implies

$$\begin{aligned}\delta_p^{\mathrm{T}}\begin{bmatrix}0 & 0\\ 0 & I_{dn_f}\end{bmatrix}\bar{H}^{\mathrm{T}}(g-g^*) &= \delta_p^{\mathrm{T}}\bar{H}^{\mathrm{T}}(g-g^*)\\ &= (p-p^*)\bar{H}^{\mathrm{T}}(g-g^*)=0. \end{aligned}\quad (24)$$

Equation (24) implies that $g=g^*$ by Lemma 2 and consequently $\delta_p=0$. ■

As shown in the proof of Theorem 2, the proportional control term in (17) converges to zero so that the geometric pattern is achieved, and the integral term converges to v_c so that each follower could reach the common constant velocity.

Theorem 2 relies on the assumption of interagent collision avoidance. We next give a sufficient condition on the position and velocity of the initial formation to simultaneously guarantee collision avoidance and system convergence. In this case, the collision-free assumption could be dropped.

Corollary 3 (Collision avoidance for single-integrator formation tracking): Under Assumption 1, there always exists a sufficiently small positive constant α such that if the Lyapunov function in (21) satisfies $V(0)\le\alpha$, then $\|p_i(t)-p_j(t)\|\ge\gamma$ for all $i,j\in\mathcal{V}$ and all $t\ge 0$ and $p(t)$ converges to $p^*(t)$ asymptotically.

Proof: With V defined in (21), inequality $V\le\alpha$ implies $e^{\mathrm{T}}(g-g^*)\le\alpha$. It then follows from (18) that

$$\frac{1}{\beta}\frac{\|\delta_p\|^2}{\|\delta_p\|+\|\tilde{p}^*\|}\le\alpha,$$

where $\beta:=2\lambda_{\min}(\mathcal{B}_{ff})/\|\bar{H}\|$. Similar to (23), the above inequality implies $\|\delta_p\|\in[0,\xi_+]$, where $\xi_+=(\alpha\beta+\sqrt{\alpha^2\beta^2+4\alpha\beta\|\tilde{p}^*\|})/2$. It is obvious that there always exists a sufficiently small α such that $\xi_+\le\epsilon$, where ϵ is given in (15), and consequently $\|\delta_p\|\le\xi_+\le\epsilon$.

At time $t=0$, since $V(0)\le\alpha$, we have $\|\delta_p(0)\|\in[0,\epsilon]$. Consequently, $\|p_i(0)-p_j(0)\|\ge\gamma$ for all i,j by (16) and hence there is no interagent collision in the initial formation. Since $\dot{V}\le 0$ in the absence of interagent collision according to (22), $\|\delta_p(t)\|$ will not escape from $[0,\epsilon]$. Otherwise, there must exist an escape time t_1 such that $V(t_1)=\alpha$ and $\dot{V}(t_1)>0$, which is impossible. As a result, $\|\delta_p(t)\|\in[0,\epsilon]$ for all $t\ge 0$ and hence collision avoidance is guaranteed. It then follows from Theorem 2 that $\|\delta_p(t)\|$ converges to zero asymptotically. ■

The integral control in (17) could also handle input disturbances as shown below.

Corollary 4 (Constant input disturbance): Under Assumptions 1 and 2, if $\dot{p}_i=u_i+\omega_i$ for $i\in\mathcal{V}_f$ where $\omega_i\in\mathbb{R}^d$ is an unknown constant disturbance and u_i is the right-hand side of the control law in (17), then $p(t)$ convergence to $p^*(t)$ asymptotically.

Proof: Denote $\omega_f\in\mathbb{R}^{dn_f}$ as the vector collecting all ω_i for $i\in\mathcal{V}_f$ and $\omega=[0,\omega_f^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{dn}$. The formation dynamics become

$$\dot{p}=-k_{\mathrm{p}}\begin{bmatrix}0 & 0\\ 0 & I_{dn_f}\end{bmatrix}\bar{H}(g-g^*)+\eta+\omega$$

$$\dot{\eta}=-k_{\mathrm{I}}\begin{bmatrix}0 & 0\\ 0 & I_{dn_f}\end{bmatrix}\bar{H}(g-g^*).$$

Define the error states as $\delta_p=p-p^*$ and $\tilde{\eta}=\eta+\omega-\mathbf{1}_n\otimes v_c$. Choose the Lyapunov function $V=e^{\mathrm{T}}(g-g^*)+\tilde{\eta}^{\mathrm{T}}\tilde{\eta}/(2k_{\mathrm{I}})$. Similar to (22), it can be shown that

$$\dot{V}=-k_{\mathrm{p}}(g-g^*)^{\mathrm{T}}\bar{H}\begin{bmatrix}0 & 0\\ 0 & I_{dn_f}\end{bmatrix}\bar{H}^{\mathrm{T}}(g-g^*)\le 0.$$

The rest proof is the same as that of Theorem 2. ■

As shown in Corollary 4, the integral term η_i in the control law eventually converges to $v_c-\omega_i$. As a result, the unknown disturbance is compensated by the integral term.

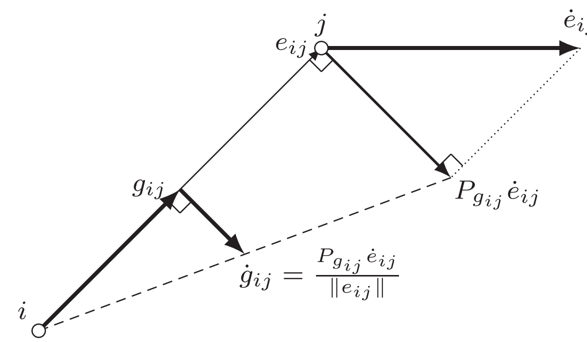

Fig. 2. Geometric relationship between $\dot{g}_{ij}$ and $\dot{e}_{ij}$.

IV. Bearing-Only Formation Tracking Control: Double-Integrator Agents

This section studies the case where each agent can be modeled as a double integrator: $\dot{p}_i(t)=v_i(t)$, $\dot{v}_i(t)=u_i(t)$, where $u_i(t)$ is the acceleration input to be designed. The leaders move at a common constant velocity v_c. For follower $i\in\mathcal{V}_f$, our proposed bearing-only control law is

$$\begin{aligned}\dot{p}_i(t)&=v_i(t)\\ \dot{v}_i(t)&=k_p\sum_{j\in\mathcal{N}_i}\left(g_{ij}(t)-g_{ij}^*\right)+k_v\sum_{j\in\mathcal{N}_i}\dot{g}_{ij}(t)\end{aligned}\quad (25)$$

where k_p and k_v are positive constant control gains. Control law (25) requires two types of measurements: The first is relative bearings $\{g_{ij}(t)\}_{j\in\mathcal{N}_i}$ and the second is the varying rate of the bearings $\{\dot{g}_{ij}(t)\}_{j\in\mathcal{N}_i}$.

The geometric interpretation of $\dot{g}_{ij}$ is illustrated in Fig. 2. The varying rate $\dot{g}_{ij}$ carries certain information of the relative velocity $\dot{e}_{ij}$, which is why $\dot{g}_{ij}$ is useful for controlling double integrators. However, $\dot{e}_{ij}$ could not be fully recovered from $\dot{g}_{ij}$. That is because the velocity magnitude and the velocity component along the direction of g_{ij} are both missing in $\dot{g}_{ij}$. As a result, an infinite number of $\dot{e}_{ij}$ could correspond to the same value of $\dot{g}_{ij}$.

In practice, if g_{ij} can be measured, the value of $\dot{g}_{ij}$ can also be measured easily. For example, if an agent uses onboard vision to measure relative bearings, the velocity of a target moving in the image can be obtained by techniques such as optical flow. The value of $\dot{g}_{ij}$ can be then calculated from optical flow based on the pin-hole camera model [23].

In order to prove the formation stability, rewrite control law (25) in a matrix form as

$$\dot{p} = v$$
$$\dot{v} = -k_p \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*) - k_v \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}} \dot{g}. \tag{26}$$

The initial values satisfy

$$p(0) = \begin{bmatrix} p_\ell^*(0) \\ p_f(0) \end{bmatrix}, \quad v(0) = \begin{bmatrix} \mathbf{1}_{dn_\ell} \otimes v_c \\ v_f(0) \end{bmatrix}$$

where $p_f(0)$ and $v_f(0)$ could be arbitrary. Define the error states as

$$\delta_p = p - p^*, \quad \delta_v = v - \mathbf{1}_n \otimes v_c.$$

The objective is to prove that $\delta_p \to 0$ and $\delta_v \to 0$ as $t \to \infty$.

Theorem 3 (Double-integrator formation tracking): Under Assumptions 1 and 2, $p(t)$ converges to $p^*(t)$ asymptotically by the action of control law (25), where $p^*(t)$ represents the target configuration moving at velocity v_c with a fixed geometric pattern as defined in Definition 1.

Proof: Since $\delta_p = [0, \delta_{p_f}^{\mathrm{T}}]^{\mathrm{T}}$ and $\delta_v = [0, \delta_{v_f}^{\mathrm{T}}]^{\mathrm{T}}$, we have

$$\delta_p^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} = \delta_p^{\mathrm{T}}, \quad \delta_v^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} = \delta_v^{\mathrm{T}}.$$

Consider the Lyapunov function

$$V = k_p e^{\mathrm{T}}(g - g^*) + \frac{1}{2}\delta_v^{\mathrm{T}} \delta_v \geq 0.$$

Similar to the Lyapunov function in (21), it can be shown that $V = 0$ if and only if $p = p^*$ and $\delta_v = 0$. Note that $\bar{H}\delta_v = \bar{H}v$ since $\bar{H}(\mathbf{1}_n \otimes v_c) = 0$. Then, the derivative of V is

$$\begin{aligned} \dot{V} &= k_p (g - g^*)^{\mathrm{T}} \bar{H}\dot{p} + \delta_v^{\mathrm{T}} \dot{\delta}_v \\ &= k_p (g - g^*)^{\mathrm{T}} \bar{H} v - k_p \delta_v^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*) \\ &\qquad - k_v \delta_v^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}} \dot{g} \\ &= k_p (g - g^*)^{\mathrm{T}} \bar{H} v - k_p \delta_v^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g - g^*) - k_v \delta_v^{\mathrm{T}} \bar{H}^{\mathrm{T}} \dot{g} \\ &= k_p (g - g^*)^{\mathrm{T}} \bar{H} v - k_p v^{\mathrm{T}} \bar{H}^{\mathrm{T}}(g - g^*) - k_v v^{\mathrm{T}} \bar{H}^{\mathrm{T}} \dot{g} \\ &= -k_v v^{\mathrm{T}} \bar{H}^{\mathrm{T}} \dot{g}. \end{aligned} \tag{27}$$

Since $\bar{H}p = e$, we have $\bar{H}v = \bar{H}\dot{p} = \dot{e}$. As a result, (27) implies

$$\dot{V} = -k_v \dot{e}^{\mathrm{T}} \dot{g} = -k_v \sum_{k=1}^{m} \dot{e}_k^{\mathrm{T}} \dot{g}_k = -k_v \sum_{k=1}^{m} \dot{e}_k^{\mathrm{T}} \frac{P_{g_k}}{\|e_k\|} \dot{e}_k \leq 0$$

where the last equality is due to (1).

Since $\dot{V} \leq 0$, $V(t) \leq V(0)$ for all t. As a result, $e^{\mathrm{T}}(g - g^*)$ and $\|\delta_v\|$ are always bounded. By the lower bound of $e^{\mathrm{T}}(g - g^*)$ in Lemma 2, $\|\delta_p\|$ is also bounded. As a result, there exists a compact set of δ_p and δ_v that is invariant under the error dynamics. By the invariance principle [22, Th. 4.4], the error states converge to the set where $\dot{V} = 0$. When $\dot{V} = 0$, $P_{g_k} \dot{e}_k = 0 \Rightarrow \dot{g}_k = 0$ and hence g is invariant. Case 1: if g is invariant and $g = g^*$, then the theorem is proved. Case 2: if g is invariant but $g \neq g^*$, the right-hand side of (26) is constant and nonzero [otherwise, $\dot{g} = 0$ implies $g = g^*$ as shown in (24)]. Consequently, $\|v\|$ will eventually increase to infinity and so does $\|\delta_v\| = \|v - v_c\|$, which contradicts the fact that $\|\delta_v\|$ is bounded. ■

The following is a sufficient condition to simultaneously guarantee collision avoidance and formation convergence. With this condition, Assumption 2 could be dropped.

Corollary 5 (Double-integrator collision avoidance): Under Assumption 1, there always exists a sufficiently small positive constant α such that if $V(0) \leq \alpha$, then $\|p_i(t) - p_j(t)\| \geq \gamma$ for all $i, j \in \mathcal{V}$ and all $t \geq 0$ and $p(t)$ converges to $p^*(t)$ asymptotically.

Proof: The proof of Corollary 5 is similar to Corollary 3 and hence omitted here. ■

V. Bearing-Only Formation Tracking Control: Unicycle Agents

This section studies bearing-only formation control of unicycle agents. Let $p_i = [x_i, y_i]^{\mathrm{T}} \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$ be the center coordinate and heading angle of agent i, respectively. The motion of agent i is governed by the unicycle model

$$\dot{x}_i = v_i \cos\theta_i \quad \dot{y}_i = v_i \sin\theta_i \quad \dot{\theta}_i = w_i \tag{28}$$

where $v_i \in \mathbb{R}$ and $w_i \in \mathbb{R}$ are the linear and angular velocities to be designed. Denote $h_i = [\cos\theta_i, \sin\theta_i]^{\mathrm{T}} \in \mathbb{R}^2$ and $h_i^\perp = [-\sin\theta_i, \cos\theta_i]^{\mathrm{T}} \in \mathbb{R}^2$. Then, the unicycle model (28) can be rewritten as

$$\dot{p}_i = v_i h_i \quad \dot{h}_i = w_i h_i^\perp. \tag{29}$$

We only consider unicycle agents moving in the plane in this paper, though model (29) could also characterize nonholonomic agents moving in three dimensions [24].

The first part of this section considers moving target formations of unicycles. The second part studies stationary target formations while the unicycle model is subject to certain motion constraints. In either part, the heading angles of the unicycles are not required to form any desired patterns and hence relative heading angles are not required to be measured.

A. Tracking Moving Target Formations

There are two conventional yet simple approaches to handle the unicycle model. The first is to convert the unicycle model to a single-integrator model by feedback linearization [25, Sec. V. A]. Then, the control law designed for single integrators in (17) could be applied. This approach is an approximation because it aims to control the motion of some offset points on the unicycles instead of their center points, whereas the bearing measurements are with respect to the center points. When the offset points are selected to be closer to the center points, the approximation would be more accurate, but the magnitude of the resultant control input may also be larger. The second approach is to convert the unicycle model to a double-integrator model by feedback linearization [26, Sec. 3]. Then, the control law designed for double integrators in (25) could be applied. The limitation of this approach is that the feedback linearization relies on an assumption that the velocity of each unicycle is never zero, which may not be guaranteed.

We propose an alternative control law based on the rigorous unicycle model. For leader $i \in \mathcal{V}_\ell$, it holds that $v_i = v_c$, $w_i = 0$, and θ_i is aligned with v_c. For follower $i \in \mathcal{V}_f$, the proposed control law is

$$v_i = h_i^{\mathrm{T}} \left[k_p \sum_{j \in \mathcal{N}_i} (g_{ij} - g_{ij}^*) + v_c \right],$$
$$w_i = (h_i^{\perp})^{\mathrm{T}} \left[k_h \sum_{j \in \mathcal{N}_i} (g_{ij} - g_{ij}^*) + v_c \right], \tag{30}$$

where k_p and k_h are two constant positive control gains. The design of this control law is inspired by [27]. One limitation of this control law is that it requires the information of v_c, which is only known by the leaders. Each follower could estimate v_c by distributed consensus protocols [28], [29]. However, distributed consensus requires wireless communication among the agents. As a comparison, the previous control laws proposed in this paper do not require wireless communication.

The formation stability under control law (30) is analyzed as follows. The following analysis does not involve the estimation of v_c. If v_c could be estimated by each follower within finite time by, for example, finite-time consensus protocols [30], [31], the following stability analysis still applies immediately.

For leader $i \in \mathcal{V}_i$, we have $\dot{p}_i = v_c$ and $\dot{h}_i = 0$. For follower i, substituting control law (30) into (29) gives

$$\dot{p}_i = h_i h_i^{\mathrm{T}} \left[k_p \sum_{j \in \mathcal{N}_i} (g_{ij} - g_{ij}^*) + v_c \right]$$
$$\dot{h}_i = h_i^{\perp} (h_i^{\perp})^{\mathrm{T}} \left[k_h \sum_{j \in \mathcal{N}_i} (g_{ij} - g_{ij}^*) + v_c \right]. \tag{31}$$

The formation stability is analyzed in the following result.

Theorem 4 (Unicycle formation tracking): Under Assumptions 1 and 2, $p(t)$ converges to $p^*(t)$ asymptotically by the action of control law (31), where $p^*(t)$ represents the target configuration moving at velocity v_c with a fixed geometric pattern as defined in Definition 1.

Proof: The matrix-vector form of the formation dynamics implied by (31) is

$$\dot{p} = -\begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} k_p \bar{H}^{\mathrm{T}} (g - g^*) + \begin{bmatrix} I_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} (\mathbf{1}_n \otimes v_c),$$
$$\dot{h} = -\begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^{\perp} (h_i^{\perp})^{\mathrm{T}}} \end{bmatrix} k_h \bar{H}^{\mathrm{T}} (g - g^*) + \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^{\perp} (h_i^{\perp})^{\mathrm{T}}} \end{bmatrix} (\mathbf{1}_n \otimes v_c) \tag{32}$$

where $D_{h_i h_i^{\mathrm{T}}} = \mathrm{blkdiag}(\dots, h_i h_i^{\mathrm{T}}, \dots) \in \mathbb{R}^{dn_f \times dn_f}$ and $D_{h_i^{\perp} (h_i^{\perp})^{\mathrm{T}}} = \mathrm{blkdiag}(\dots, h_i^{\perp} (h_i^{\perp})^{\mathrm{T}}, \dots) \in \mathbb{R}^{dn_f \times dn_f}$ where $i \in \mathcal{V}_f$. The initial value of p satisfies $p(0) = [p_\ell^*(0)^{\mathrm{T}}, p_f(0)^{\mathrm{T}}]^{\mathrm{T}}$ where $p_f(0)$ could be arbitrary. The initial heading $h_i(0)$ of leader $i \in \mathcal{V}_\ell$ should be consistent with v_c: if $v_c \neq 0$ then $h_i(0)$ should be parallel with v_c; otherwise, if $v_c = 0$ then $h_i(0)$ could be arbitrary. The initial heading of a follower could be arbitrary.

Consider the Lyapunov function

$$V = e^{\mathrm{T}} (g - g^*) + \frac{1}{2k_h} \| h - \mathbf{1}_n \otimes v_c \|^2 \geq 0. \tag{33}$$

The global minimum value of V is $n(1 - \|v_c\|)^2 / (2k_h)$. If $\|v_c\| = 0$, this value is reached when $\delta_p = 0$ and h_i is arbitrary. If $\|v_c\| \neq 0$, this value is reached when $\delta_p = 0$ and $h_i = v_c / \|v_c\|$ for all i. The derivative of V along the system trajectory is

$$\begin{aligned}
\dot{V} &= (g - g^*)^{\mathrm{T}} \bar{H} \dot{p} + \frac{1}{k_h} (h - \mathbf{1}_n \otimes v_c)^{\mathrm{T}} \dot{h} \\
&= -(g - g^*)^{\mathrm{T}} \bar{H} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} k_p \bar{H}^{\mathrm{T}} (g - g^*) \\
&\quad + (g - g^*)^{\mathrm{T}} \bar{H} \begin{bmatrix} I_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} (\mathbf{1}_n \otimes v_c) \\
&\quad - (h - \mathbf{1}_n \otimes v_c)^{\mathrm{T}} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^{\perp} (h_i^{\perp})^{\mathrm{T}}} \end{bmatrix} \bar{H}^{\mathrm{T}} (g - g^*) \\
&\quad + \frac{1}{k_h} (h - \mathbf{1}_n \otimes v_c)^{\mathrm{T}} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^{\perp} (h_i^{\perp})^{\mathrm{T}}} \end{bmatrix} \mathbf{1}_n \otimes v_c \\
&= -k_p (g - g^*)^{\mathrm{T}} \bar{H} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} \bar{H}^{\mathrm{T}} (g - g^*) \\
&\quad + (g - g^*)^{\mathrm{T}} \bar{H} \begin{bmatrix} I_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} (\mathbf{1}_n \otimes v_c)
\end{aligned}$$

$$+ (\mathbf{1}_n \otimes v_c)^{\mathrm{T}} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^\perp (h_i^\perp)^{\mathrm{T}}} \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*)$$
$$- \frac{1}{k_h}(\mathbf{1}_n \otimes v_c)^{\mathrm{T}} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^\perp (h_i^\perp)^{\mathrm{T}}} \end{bmatrix} (\mathbf{1}_n \otimes v_c) \quad (34)$$

where the last equality is due to $h_i^{\mathrm{T}}(h_i^\perp (h_i^\perp)^{\mathrm{T}}) = 0$. Since $h_i h_i^{\mathrm{T}} + h_i^\perp (h_i^\perp)^{\mathrm{T}} = I_d$, the sum of the second and third items in (34) is $(g - g^*)^{\mathrm{T}} \bar{H}(\mathbf{1} \otimes v_c)$ which equals zero. As a result,

$$\dot{V} = -k_p (g - g^*)^{\mathrm{T}} \bar{H} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*)$$
$$- \frac{1}{k_h}(\mathbf{1}_n \otimes v_c)^{\mathrm{T}} \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^\perp (h_i^\perp)^{\mathrm{T}}} \end{bmatrix} (\mathbf{1}_n \otimes v_c) \leq 0.$$

Since $\dot{V} \leq 0$, $V(t) \leq V(0)$ for all t. As a result, $e^{\mathrm{T}}(g - g^*)$ is always bounded. By the lower bound of $e^{\mathrm{T}}(g - g^*)$ in Lemma 2, $\|\delta_p\|$ is also always bounded. As a result, there exists a compact set of δ_p and $h_i - v_c$ that is invariant under the dynamics. By the invariance principle [22, Th. 4.4], the states converge to the set where $\dot{V} = 0$. It follows from $\dot{V} = 0$ that

$$\begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i h_i^{\mathrm{T}}} \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*) = 0 \quad (35)$$

$$\begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & D_{h_i^\perp (h_i^\perp)^{\mathrm{T}}} \end{bmatrix} (\mathbf{1}_n \otimes v_c) = 0. \quad (36)$$

Since $h_i h_i^{\mathrm{T}} + h_i^\perp (h_i^\perp)^{\mathrm{T}} = I_d$, substituting (35) and (36) into (32) gives

$$\dot{p} = \mathbf{1}_n \otimes v_c \quad (37)$$

$$\dot{h} = -k_h \begin{bmatrix} 0_{dn_\ell} & 0 \\ 0 & I_{dn_f} \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*). \quad (38)$$

Equation (37) indicates that all the agents move at the same common velocity v_c. As a result, the relative positions of the agents are time-invariant and consequently $\bar{H}(g - g^*)$ is constant. Assume that $\dot{h}$ in (38) is nonzero. Then, for certain $i \in \mathcal{V}_f$, h_i is time-varying, which is impossible to have (35). As a result, $\dot{h} = 0$ and it follows from (24) that $g = g^*$. ■

According to the proof of Theorem 4, if the leader velocity v_c is nonzero, then the heading of every agent would become aligned with v_c eventually; otherwise, the heading is unspecified in the final formation.

B. Formation Stabilization Subject to Constraints

We next present a bearing-only formation control law to handle unicycles subject to velocity saturation and certain path constraints. This control law is applicable to stationary target formations and it will be our future work to study tracking moving formations subject constraints.

Suppose v_i and w_i are constrained by

$$-\mathrm{v}_i^{\mathrm{b}} \leq v_i \leq \mathrm{v}_i^{\mathrm{f}}, \quad -\mathrm{w}_i^{\mathrm{r}} \leq w_i \leq \mathrm{w}_i^{\mathrm{l}},$$

where $\mathrm{v}_i^{\mathrm{f}}, \mathrm{v}_i^{\mathrm{b}} > 0$ are the maximum forward and backward linear speeds, respectively. The constants $\mathrm{w}_i^{\mathrm{r}}, \mathrm{w}_i^{\mathrm{l}} > 0$ are the maximum left-turn and right-turn angular speeds, respectively. Here $v_i > 0$ means the agent moves forward, and $v_i < 0$ backward; and $w_i > 0$ means the agent turns its heading vector to the left (i.e., counterclockwise), and $w_i < 0$ to the right (i.e., clockwise). Define the saturation functions for the linear and angular speeds for agent i as

$$\mathrm{sat}_{v_i}(x) = \begin{cases} -\mathrm{v}_i^{\mathrm{b}}, & x \in (-\infty, -\mathrm{v}_i^{\mathrm{b}}) \\ x, & x \in [-\mathrm{v}_i^{\mathrm{b}}, \mathrm{v}_i^{\mathrm{f}}] \\ \mathrm{v}_i^{\mathrm{f}}, & x \in (\mathrm{v}_i^{\mathrm{f}}, +\infty) \end{cases}$$

$$\mathrm{sat}_{w_i}(x) = \begin{cases} -\mathrm{w}_i^{\mathrm{r}}, & x \in (-\infty, -\mathrm{w}_i^{\mathrm{r}}) \\ x, & x \in [-\mathrm{w}_i^{\mathrm{r}}, \mathrm{w}_i^{\mathrm{l}}] \\ \mathrm{w}_i^{\mathrm{l}}, & x \in (\mathrm{w}_i^{\mathrm{l}}, +\infty). \end{cases}$$

The saturation bounds $\mathrm{v}_i^{\mathrm{f}}, \mathrm{v}_i^{\mathrm{b}}, \mathrm{w}_i^{\mathrm{r}}, \mathrm{w}_i^{\mathrm{l}}$ may vary for different agents.

The proposed bearing-only formation control law for unicycle $i \in \mathcal{V}_f$ is

$$v_i = \mathrm{sat}_{v_i}\left\{\alpha_i(t) h_i^{\mathrm{T}} f_i\right\}$$
$$w_i = \mathrm{sat}_{w_i}\left\{(h_i^\perp)^{\mathrm{T}} h_i^{\mathrm{d}}(t)\right\} \quad (39)$$

where

$$f_i = \sum_{j \in \mathcal{N}_i} (g_{ij} - g_{ij}^*). \quad (40)$$

The time-varying quantities $\alpha_i(t)$ and $h_i^{\mathrm{d}}(t)$ provide additional freedom to control agent i. More specifically, $\alpha_i(t)$ is a time-varying positive scalar. It can be designed to adjust the linear velocity magnitude so as to slow down or speed up each agent if needed. The vector $h_i^{\mathrm{d}}(t) \in \mathbb{R}^2$ represents the desired heading vector for unicycle i. The angular speed control in (39) aims to turn h_i to align with h_i^{d}. As a result, h_i^{d} can be designed to adjust the heading of agent i so as to satisfy certain motion constraints such as avoiding obstacles. When there are no obstacles or path constraints, it can be simply chosen as $h_i^{\mathrm{d}} = f_i$.

The stability analysis of the control law is given below.

Theorem 5 (Unicycle formation subject to constraints): Under Assumptions 1 and 2, control law (39) drives $p(t)$ to p^* asymptotically, where p^* is the stationary target configuration, if the variables $\alpha_i(t)$ and $h_i^{\mathrm{d}}(t)$ satisfy the following conditions:

1) $\alpha_i(t)$ is uniformly continuous in t and bounded as $0 < \alpha_{\min} \leq \alpha_i(t) \leq \alpha_{\max}$;
2) $0 \leq \phi_i(t) \leq \phi_{\max} < \pi/2$ where $\phi_i(t)$ is the angle between $h_i^{\mathrm{d}}(t)$ and f_i;
3) $\|h_i^{\mathrm{d}}(t)\| = 0$ if and only if $\|f_i\| = 0$.

Proof: The stability analysis is similar to [32, Th. 3–4]. We merely outline the important steps in the stability analysis as below. Substituting control law (39) into the unicycle model in (29) gives

$$\dot{p}_i = h_i \mathrm{sat}_{v_i}(\alpha_i h_i^{\mathrm{T}} f_i)$$
$$\dot{h}_i = h_i^\perp \mathrm{sat}_{w_i}((h_i^\perp)^{\mathrm{T}} h_i^{\mathrm{d}}), \quad i \in \mathcal{V}_f. \quad (41)$$

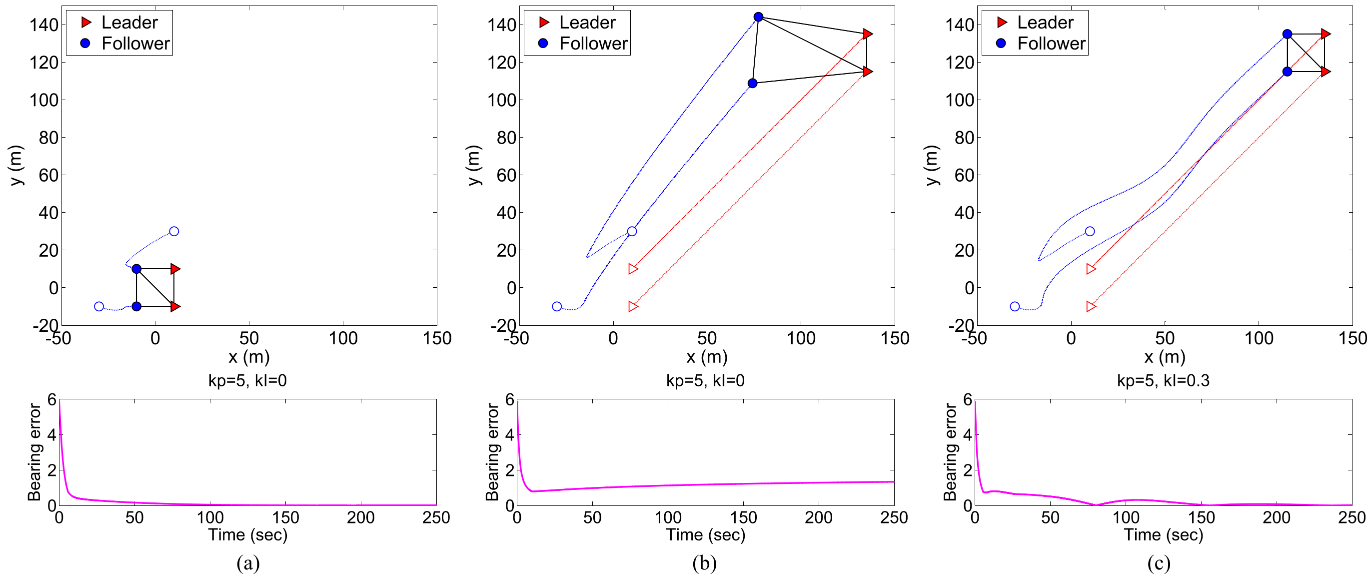

Fig. 3. Simulation results of control law (17) designed for single-integrator agents. (a) The leaders are stationary and the integral control is not used. (b) The leaders move at a constant common velocity and the integral control is not used. (c) The leaders are moving and the integral control is used. The hollow markers represent the initial positions of the agents. The bearing error is $\sum_{(i,j)\in\mathcal{E}} \|g_{ij} - g_{ij}^*\|$.

First of all, rewrite the saturation function as $\mathrm{sat}_{v_i}(\alpha_i h_i^{\mathrm{T}} f_i) = \kappa_i \alpha_i h_i^{\mathrm{T}} f_i$, where

$$\kappa_i = \begin{cases} \dfrac{\mathrm{v}_i^{\mathrm{b}}}{-\alpha_i h_i^{\mathrm{T}} f_i}, & \alpha_i h_i^{\mathrm{T}} f_i \in (-\infty, -\mathrm{v}_i^{\mathrm{b}}) \\ 1, & \alpha_i h_i^{\mathrm{T}} f_i \in [-\mathrm{v}_i^{\mathrm{b}}, \mathrm{v}_i^{\mathrm{f}}] \\ \dfrac{\mathrm{v}_i^{\mathrm{f}}}{\alpha_i h_i^{\mathrm{T}} f_i}, & \alpha_i h_i^{\mathrm{T}} f_i \in (\mathrm{v}_i^{\mathrm{f}}, +\infty). \end{cases} \quad (42)$$

With the notation of κ_i, the control law in (41) can be rewritten as $\dot{p}_i = \kappa_i \alpha_i h_i h_i^{\mathrm{T}} f_i$. Then, the matrix-vector form of the control law is

$$\dot{p} = \begin{bmatrix} \dot{p}_\ell \\ \dot{p}_f \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*)$$

where $D = \mathrm{blkdiag}(\kappa_{n_\ell+1}\alpha_{n_\ell+1} h_{n_\ell+1} h_{n_\ell+1}^{\mathrm{T}}, \ldots, \kappa_n \alpha_n h_n h_n^{\mathrm{T}})$ is a $(dn_f) \times (dn_f)$ positive semidefinite block diagonal matrix.

Consider the Lyapunov function $V = e^{\mathrm{T}}(g - g^*)$. Since $e^T \dot{g} = 0$, the time derivative of V is

$$\begin{aligned} \dot{V} &= (g - g^*)^{\mathrm{T}} \dot{e} = (g - g^*)^{\mathrm{T}} \bar{H} \dot{p} \\ &= -(g - g^*)^{\mathrm{T}} \bar{H} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \bar{H}^{\mathrm{T}}(g - g^*) \\ &= -\sum_{i\in\mathcal{V}_f} \kappa_i \alpha_i f_i^{\mathrm{T}} h_i h_i^{\mathrm{T}} f_i \leq 0. \end{aligned}$$

Since V is nonincreasing and bounded from below, V converges as $t \to \infty$. The next step is to prove that $\dot{V}$ is uniformly continuous in t by showing that h_i, f_i, and κ_i are uniformly continuous in t. The rest of the proof is similar to [32, Th. 3–4] and omitted here. ■

The conditions on $\alpha_i(t)$ and $h_i^{\mathrm{d}}(t)$ in Theorem 5 are mild. In particular, α_i may vary within a wide interval. The heading of h_i^{d} can vary freely as long as the angle between h_i^{d} and f_i is less than $\pi/2$. Note that h_i^{d} is not required to be continuous. These mild conditions provide more freedom for the agents to fulfil motion constraints without jeopardizing formation stability. Experimental results will be given later to demonstrate how to properly design h_i^{d} to satisfy motion constraints such as obstacle avoidance.

VI. Simulation and Experimental Results

A. Simulation for Single-Integrator Agents

Fig. 3 shows simulation examples to demonstrate control law (17) designed for single-integrator agents. The target formation is a square with two leaders as shown in Fig. 1(b).

Three relevant simulation scenarios are studied. In the first scenario, the leaders are stationary. As shown in Fig. 3(a), control law (17) is able to steer the agents to the target formation without the integral control term (i.e., $k_{\mathrm{I}} = 0$). In the second scenario, the leaders move at a common nonzero constant velocity. In this case, without the integral control term, control law (17) is not able to track the moving target formation as shown in Fig. 3(b), and the position-tracking errors diverge to infinity. In the third scenario, with the integral control term, the control law can successfully track the moving target formation as shown in Fig. 3(c). The control gains are selected as $k_{\mathrm{p}} = 5$ and $k_{\mathrm{I}} = 0.3$. It is observed in the simulation that large k_{p} could accelerate formation convergence, but also leads to large velocity input. Large k_{I} could accelerate formation convergence, but may also lead to trajectory oscillation.

B. Simulation for Double-Integrator Agents

Fig. 4 shows a simulation example to demonstrate control law (25) designed for double-integrator agents. The target formation is the three-dimensional (3-D) cube with two leaders as

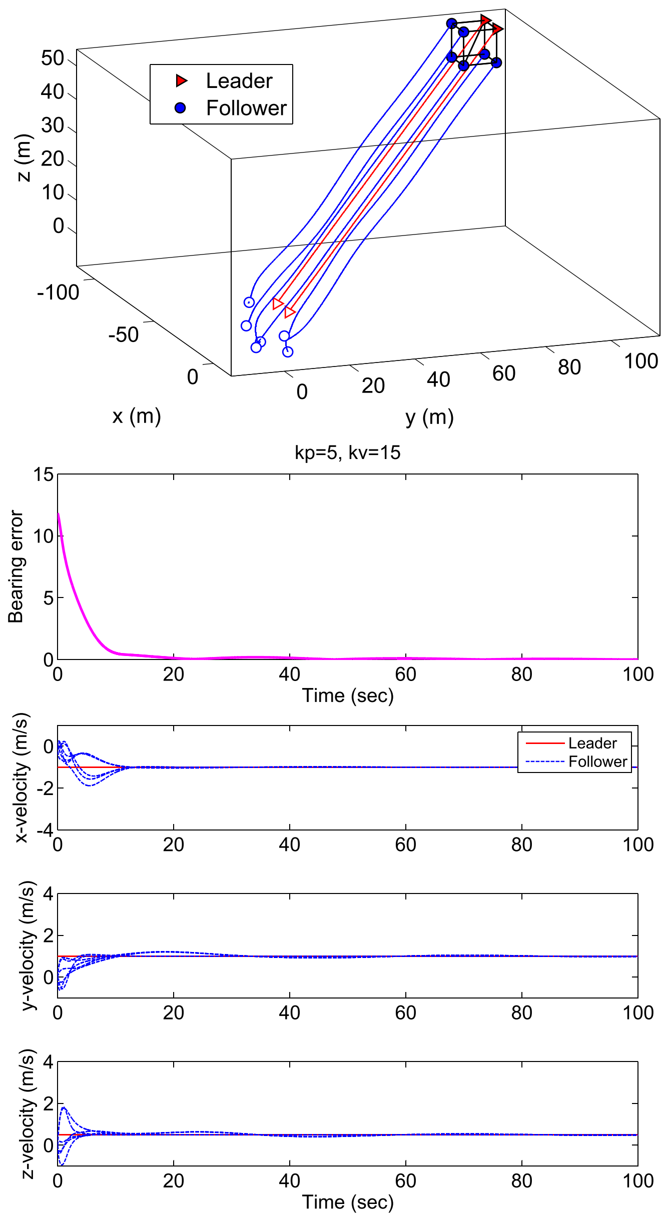

Fig. 4. Simulation results for control law (25) designed for double-integrator agents. The hollow markers represent the initial positions of the agents. The bearing error is $\sum_{(i,j)\in\mathcal{E}} \|g_{ij} - g_{ij}^*\|$.

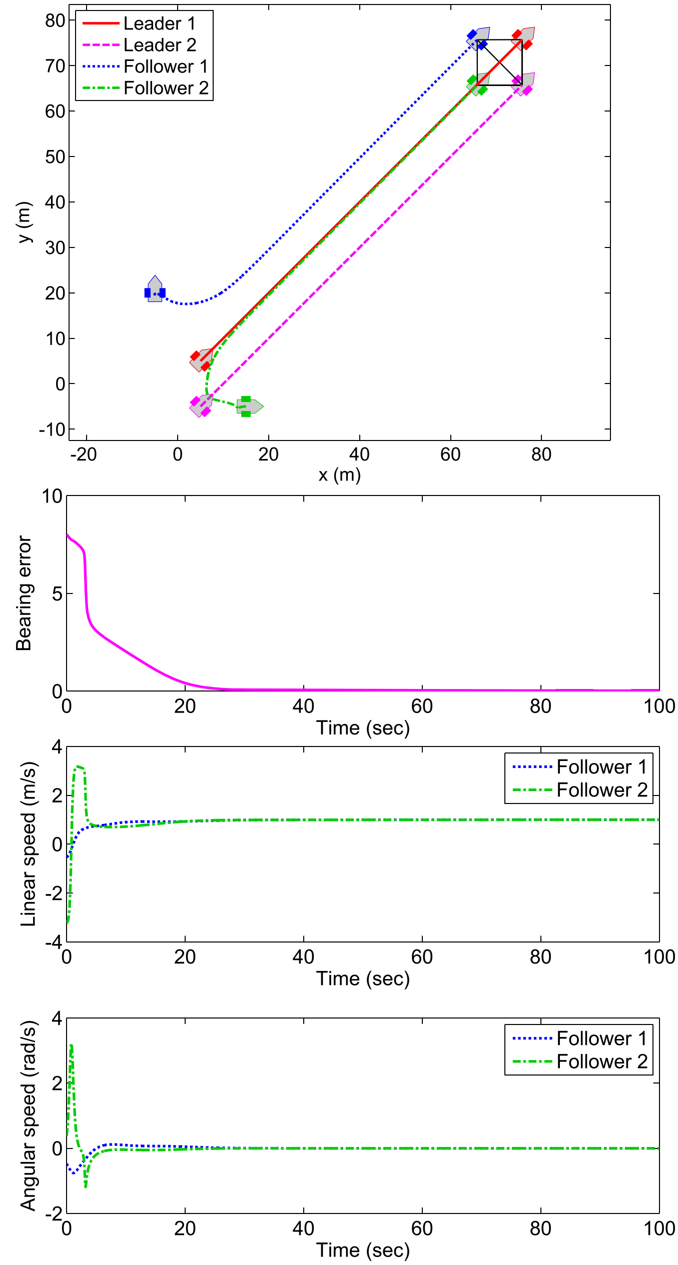

Fig. 5. Simulation results to demonstrate control law (30) designed for unicycle agents.

shown in Fig. 1(c). The two leaders move at a common nonzero constant velocity. As can be seen, the formation configuration converges to the desired one and the velocity of each follower also converges to the leaders' velocity. In the simulation, the control gains are selected as $k_p = 5$ and $k_v = 15$. According to the simulation, it is observed that too small k_v or too large k_p could lead to trajectory oscillation.

C. Simulation for Unicycles

Fig. 5 shows a simulation example to demonstrate control law (30) designed for unicycle agents. The target formation is a square with two leaders as shown in Fig. 1(b). The two leaders move at a constant velocity $v_c = [\sqrt{2}/2, \sqrt{2}/2]^{\mathrm{T}}$. As can be seen, the formation configuration converges to the desired one, the velocity of each follower converges to v_c, and the heading of each follower becomes aligned with v_c eventually. In the simulation, the control gains are selected as $k_p = k_h = 1.5$. According to the simulation, it is observed that large k_p and k_h would accelerate convergence, but may lead to large velocity input value.

D. Experiment for Unicycles

Control law (39) has been implemented and verified on real unicycle robots. The unicycle robots used in the experiment are shown in Fig. 6(a). Each robot has two wheels. The pattern on the top of each robot is used to localize the robot by a vision system. The location of each robot is estimated in a central

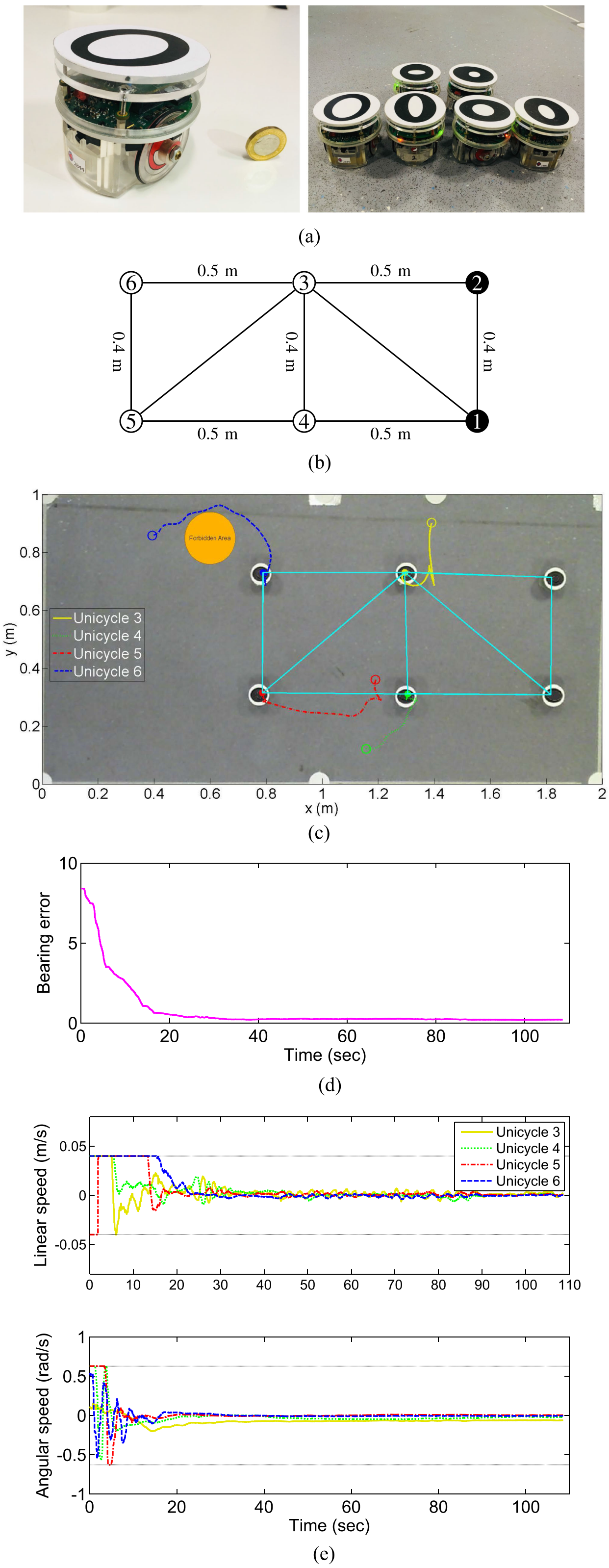

Fig. 6. Experimental results of control law (39) designed for unicycle robots subject to motion constraints. (a) Unicycle robots used in the experiments. (b) Target formation in the experiment. Agent 1 and 2 are leaders. The rest are followers. (c) Trajectories and final formation plotted against an image taken by the downward-looking camera on the ceiling. Hollow dots represent the initial positions of the robots. (d) Total bearing error $\sum_{(i,j)\in\mathcal{E}} \|g_{ij} - g^*_{ij}\|$. (e) Linear and angular velocities.

computer and then transmitted to each agent via Bluetooth. The control law is executed on each robot in a distributed manner.

In this experiment, the target formation of the six robots is given in Fig. 6(b). As shown in Fig. 6(c)–(e), the target formation is successfully achieved in the presence of an forbidden area. The coordinate of the center of the forbidden area is (0.6, 0.85) m and its radius is 0.1 m.

The obstacle-avoidance strategy used in the experiment is descried as below. When f_i in (40) does not point into the forbidden area, h_i^d is chosen as f_i. When f_i points into the forbidden area, the obstacle- avoidance mechanism is triggered and h_i^d is chosen as a unit vector pointing to the leftmost point of the forbidden area. As a result, the unicycle could pass by the left-hand side of the forbidden area as shown in Fig. 6(c). Of course, h_i^d could also be chosen to point into any other direction that does not pass through the forbidden area. In the experiment, there is merely one single round forbidden area. In this simple scenario, it is always possible to properly select h_i^d so that condition 2) in Theorem 5 is satisfied and hence formation stability is guaranteed. However, in some complicated scenarios with multiple and irregular obstacles, it may be impossible to find h_i^d to satisfy condition 2) to avoid all the obstacles.

It is worth mentioning that the formation task was successfully achieved in the presence of many practical problems in the experiment. First, since each unicycle has merely two wheels, the front bottom end or the rear bottom end of the robot always contacts the ground floor, which causes strong frictional disturbances. Second, the low-level velocity control for each robot is an open loop control, which is not able to track velocity references accurately. Third, there is a significant time delay caused by vision processing and data transmission via Bluetooth.

VII. Conclusion

This paper proposed novel bearing-only control laws to handle moving target formations and a variety of agent models including single integrators, double integrators, and unicycles. The proposed control laws are an important step towards the application of bearing-only formation control in practical tasks. In the future, this study may be generalized in several directions. First, more complicated agent dynamics, such as general linear systems [33] and directed sensing graphs [16] could be studied. Second, the bearing vectors considered in this study are expressed in a common reference frame. It is important to study how to achieve formation control using locally measured bearings. Third, more sophisticated collision avoidance strategies, such as reciprocal velocity obstacle [34] could be employed to achieve avoidance of dynamical obstacles.

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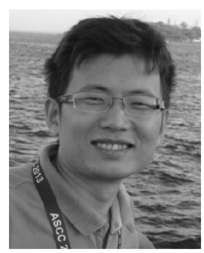

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