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RESEARCH ARTICLE

Finite-time Stabilization of Cyclic Formations using Bearing-only Measurements

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This paper studies decentralized formation control of multiple vehicles in the plane when each vehicle can only measure the local bearings of their neighbors by using bearing-only sensors. Since the inter-vehicle distance cannot be measured, the target formation involves no distance constraints. More specifically, the target formation considered in this paper is an angle-constrained cyclic formation, where each vehicle has exactly two neighbors and the angle at each vehicle subtended by its two neighbors is pre-specified. To stabilize the target formation, we propose a discontinuous control law that only requires the sign information of the angle errors. Due to the discontinuity of the proposed control law, the stability of the closed-loop system is analyzed by employing a locally Lipschitz Lyapunov function and nonsmooth analysis tools. We prove that the target formation is locally finite-time stable with collision avoidance guaranteed.

Keywords: formation control; bearing-only measurement; discontinuous dynamic system; finite-time stability

1 Introduction

In the existing work on multi-vehicle formation control, it is commonly assumed that each vehicle can obtain the positions of their neighbors (Lin et al. 2005, Dimarogonas and Johansson 2010, Ren and Cao 2011, Tian and Wang 2013, Kopeikin et al. 2013, Keller et al. 2013, Hu et al. 2013). It is notable that *position* information essentially consists of two kinds of partial information: *bearing* and *distance*. In recent years, formation control using bearing-only (Moshtagh et al. 2008, Basiri et al. 2010, Eren 2012, Franchi and Giordano 2012) or distance-only (Cao et al. 2011) measurements has attracted some attention. In this paper we will investigate *formation control using bearing-only measurements* (or called *bearing-based formation control*). We assume that each vehicle can only measure the bearings of their neighbors and inter-vehicle distances are unavailable. In practical applications, bearings can be conveniently measured by using monocular cameras which are very common sensors nowadays. A camera inherently is a bearing-only sensor. As long as the target can be localized in the image, the bearing of the target relative to the camera can be easily calculated based on the pin-hole camera model (Ma et al. 2004, Section 3.3). Hence vision-based cooperative control tasks (Das et al. 2002, Mariottini et al. 2009) are the potential applications of the research on bearing-based formation control.

A number of interesting and challenging problems arise when only bearings are available for formation control. One key problem is how to utilize the bearing measurements for formation control. There are generally two approaches. The first approach is to implement control laws directly using bearing measurements; the second approach is to estimate vehicle positions using bearing measurements, and then implement formation control laws based on the estimated

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positions. In this paper we will adopt the first approach. The reason we do not use the second approach is that position estimation using bearings requires certain observability conditions. For example, suppose vehicle A is stationary and vehicle B can measure the bearings of vehicle A from two different angles. Then vehicle A can be localized as the intersection of the two bearings measured by vehicle B. But if vehicle B is only able to measure vehicle A from one single angle, the distance between the two vehicles is impossible to be recovered. In other words, relative motions between vehicles are fundamentally necessary for position estimation using bearing measurements. However, in many formation control tasks, the relative positions of the vehicles in the formation are required to be stationary and hence the observability condition is not satisfied.

If each vehicle is controlled based only on bearings, the inter-vehicle distance in the formation would be uncontrollable. As a result, any bearing-based formation control law can only stabilize *bearing-constrained* target formations. The term bearing-constrained as used here refers to the bearings of the edges connecting vehicles or the angles at each vehicle subtended by their neighbors are constrained. Another challenging problem is collision avoidance, which is important for all kinds of formation control problems. Collision avoidance is particularly important for bearing-based formation control as inter-vehicle distances are unmeasurable and uncontrollable. In order to make any bearing-based formation control law practically applicable, we need to guarantee collision avoidance between any vehicles no matter they are neighbors or not.

As a relatively new research topic, bearing-based formation control has not been completely solved yet. Only a few special cases have been analyzed in the literature. Moshtagh et al. (2008) proposed a distributed control law for balanced circular formations of unit speed vehicles. The proposed control law can globally stabilize balanced circular formations using bearing-only measurements. Basiri et al. (2010) studied distributed control of triangular formations of three vehicles using bearing-only measurements. The global stability of the proposed formation control law is proved by employing the Poincare-Bendixson theorem. But the Poincare-Bendixson theorem is only applicable to the scenarios involving only three or four vehicles. Eren (2012) investigated formation shape control using bearing measurements. Bearing rigidity is proposed to formulate bearing-based formation control problems. A bearing-based control law is designed for a formation of three nonholonomic vehicles. Bishop (2011) proposed a control law that can stabilize general bearing-constrained formations. However, the proposed control law in (Bishop 2011) still requires position measurements. Based on the concept of parallel rigidity, Franchi and Giordano (2012) proposed a distributed control law to stabilize bearing-constrained formations using bearing-only measurements. However, the proposed control law in (Franchi and Giordano 2012) requires communications among the vehicles. That is different from the problem considered in this paper where we assume there are no communications between any vehicles and each vehicle cannot share their bearing measurements with their neighbors.

In this paper, we study distributed bearing-based control of cyclic formations in the plane. The sensing graph of the formation is an undirected cycle with fixed topology. In the target formation, the angle at each vehicle subtended by its two neighbors is constrained. For cyclic formations the angle constraints cannot specify a unique formation shape. To well define a formation shape using angle constraints, we may assign more complicated underlying graphs such as rigid graphs to the formation. But we only consider cycle graphs in this paper and leave more complicated cases for future research. The main contributions of this work are summarized as below.

1) To stabilize angle-constrained cyclic formations, we propose a distributed discontinuous control law that only requires the sign information of the angle errors. Compared to the existing work in (Basiri et al. 2010), our control law is able to stabilize cyclic formations with an arbitrary number of vehicles. Additionally, this work requires no parallel rigidity assumptions (Bishop 2011, Eren 2012) on the target formation .
2) Finite-time control has attracted much attention in recent years (Chen et al. 2011, Meng and Lin 2012, Hong et al. 2010, 2002, Xiao et al. 2009). Besides fast convergence, finite-time control can also bring benefits such as disturbance rejection and robustness against uncertainties

(Bhat and Bernstein 2000). In this work we prove the proposed control law ensures local finite-time convergence of the angle errors. The finite-time stability of the nonlinear closed-loop system is proved by using nonsmooth analysis tools (Filippov 1988, Clarke 1983, Paden and Sastry 1987, Bacciotti and Ceragioli 1999, Cortés and Bullo 2005, Cortés 2008).

3) Collision avoidance is a particularly important issue for bearing-based formation control as inter-vehicle distances are unmeasurable. We prove that the proposed control law guarantees collision avoidance between any vehicles (no matter they are neighbors or not) given sufficiently small initial angle errors.

The paper is organized as follows. Preliminaries to graph theory and nonsmooth analysis are introduced in Section 2. Section 3 presents the problem formulation and the proposed control law. The formation stability by the proposed control law is proved in Section 4. Section 5 shows simulation results to verify the theoretical analysis. Conclusions are drawn in Section 6.

2 Preliminaries to Graph Theory and Nonsmooth Analysis

2.1 *Notations*

Given a symmetric positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$, the eigenvalues of A are denoted as $0 \le \lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$. Let $\mathbf{1} = [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$, and I be the identity matrix with appropriate dimensions. Denote $|\cdot|$ as the absolute value of a real number, and $\|\cdot\|$ as the Euclidean norm of a vector. Denote $\mathrm{Null}(\cdot)$ as the right null space of a matrix. Let $[\cdot]_{ij}$ be the entry at the ith row and jth column of a matrix, and $[\cdot]_i$ be the ith entry of a vector. Given a set S, denote $\overline{S}$ as its closure. For any angle $\alpha \in \mathbb{R}$,

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \in \mathbb{R}^{2\times 2} \tag{1}$$

is a rotation matrix satisfying $R^{-1}(\alpha) = R^{\mathrm{T}}(\alpha) = R(-\alpha)$. Geometrically, $R(\alpha)$ rotates a vector in $\mathbb{R}^2$ counterclockwise through an angle α about the origin.

2.2 *Graph Theory*

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a vertex set $\mathcal{V} = \{1, \ldots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. If $(i, j) \in \mathcal{E}$, then i and j are called to be adjacent. The set of neighbors of vertex i is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. A graph is undirected if each $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$, otherwise the graph is directed. A path from i to j in a graph is a sequence of distinct nodes starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of graph $\mathcal{G}$, then $\mathcal{G}$ is said to be connected. An undirected cycle graph is a connected graph where every vertex has exactly two neighbors.

An incidence matrix of a directed graph is a matrix E with rows indexed by edges and columns indexed by vertices[1]. Suppose (j, k) is the ith edge. Then the entry of E in the ith row and kth column is 1, the one in the ith row and jth column is -1, and the others in the ith row are zero. By definition, we have $E\mathbf{1} = 0$. If a graph is connected, the corresponding E has rank $n-1$ (see (Godsil and Royle 2001, Theorem 8.3.1)). Then $\mathrm{Null}(E) = \mathrm{span}\{\mathbf{1}\}$.

[1] In some literature such as Godsil and Royle (2001), the rows of an incidence matrix are indexed by vertices and the columns are indexed by edges.

2.3 *Nonsmooth Stability Analysis*

Next we introduce some useful concepts and facts regarding discontinuous dynamic systems (Filippov 1988, Clarke 1983, Paden and Sastry 1987, Bacciotti and Ceragioli 1999, Cortés and Bullo 2005, Cortés 2008).

2.3.1 Filippov Differential Inclusion

Consider the dynamic system

$$\dot{x}(t) = f\left(x(t)\right), \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable and essentially locally bounded function. The Filippov differential inclusion (Filippov 1988) associated with the system (2) is

$$\dot{x} \in \mathcal{F}[f](x), \tag{3}$$

where $\mathcal{F}[f] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$\mathcal{F}[f](x) = \bigcap_{r>0} \bigcap_{\mu(S)=0} \overline{\mathrm{co}} \left\{ f\left(B(x,r) \setminus S\right)\right\}. \tag{4}$$

In (4), $\overline{\mathrm{co}}$ denotes convex closure, $B(x,r)$ denotes the open ball centered at x with radius $r > 0$, and $\mu(S) = 0$ means that the Lebesgue measure of the set S is zero. The set-valued map $\mathcal{F}[f]$ associates each point x with a set. Note $\mathcal{F}[f](x)$ is multiple valued only if $f(x)$ is discontinuous at x.

A Filippov solution of (2) on $[0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [0, t_1] \to \mathbb{R}^n$ that satisfies (3) for almost all $t \in [0, t_1]$. If $f(x)$ is measurable and essentially locally bounded, the existence of Filippov solutions can be guaranteed (Cortés and Bullo 2005, Lemma 2.5) (Cortés 2008, Proposition 3) though the uniqueness cannot. The interested reader is referred to (Cortés 2008, p. 52) for the uniqueness conditions of Filippov solutions. A solution is called maximal if it cannot be extended forward in time. A set Ω is said to be weakly invariant (respectively strongly invariant) for (2), if for each $x(0) \in \Omega$, Ω contains at least one maximal solution (respectively all maximal solutions) of (2).

2.3.2 Generalized Gradient

Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function. If $V(x)$ is differentiable at x, denote $\nabla V(x)$ as the gradient of $V(x)$ with respect to x. Let M_V be the set where $V(x)$ fails to be differentiable. The generalized gradient (Clarke 1983, Cortés and Bullo 2005, Cortés 2008) of $V(x)$ is defined as

$$\partial V(x) = \mathrm{co} \left\{ \lim_{i \to +\infty} \nabla V(x_i) \mid x_i \to x,\, x_i \notin S \cup M_V \right\},$$

where co denotes convex hull and S is an arbitrary set of Lebesgue measure zero. The generalized gradient is a set-valued map. If $V(x)$ is continuously differentiable at x, then $\partial V(x) = \{\nabla V(x)\}$.

Given any set $S \subseteq \mathbb{R}^n$, let $\mathrm{Ln} : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$ be the set-valued map that associates S with the set of least-norm elements of $\overline{S}$. If S is convex, $\mathrm{Ln}(S)$ is singleton. In this paper, we only apply Ln to generalized gradients which are always convex. For a locally Lipschitz function $V(x)$, $\mathrm{Ln}(\partial V) : \mathbb{R}^n \to \mathbb{R}^n$ is called the generalized gradient vector field. The following fact (Cortés 2008, Proposition 8)

$$\mathcal{F}\left[\mathrm{Ln}\left(\partial V(x)\right)\right] = \partial V(x) \tag{5}$$

will be very useful in our work. A point x is called a critical point if $0 \in \partial V(x)$. For a critical point x, it is obvious that $\mathrm{Ln}(\partial V(x)) = \{0\}$.

2.3.3 Set-valued Lie Derivative

The evolution of a locally Lipschitz function $V(x)$ along the solutions to the differential inclusion $\dot{x} \in \mathcal{F}[f](x)$ can be characterized by the set-valued Lie derivative (Bacciotti and Ceragioli 1999, Cortés and Bullo 2005, Cortés 2008), which is defined by

$$\widetilde{\mathcal{L}}_{\mathcal{F}} V(x) = \left\{ \ell \in \mathbb{R} \mid \exists \xi \in \mathcal{F}[f](x),\ \forall \zeta \in \partial V(x),\ \xi^{\mathrm{T}} \zeta = \ell \right\}.$$

With a slight abuse of notation, we also denote $\widetilde{\mathcal{L}}_f V(x) = \widetilde{\mathcal{L}}_{\mathcal{F}} V(x)$. The set-valued Lie derivative may be empty. When $\widetilde{\mathcal{L}}_{\mathcal{F}} V(x) = \emptyset$, we take $\max \widetilde{\mathcal{L}}_{\mathcal{F}} V(x) = -\infty$ (see (Bacciotti and Ceragioli 1999, Cortés and Bullo 2005, Cortés 2008)).

A function $V : \mathbb{R}^n \to \mathbb{R}$ is called regular (Cortés 2008, p. 57) at x if the right directional derivative of $V(x)$ at x exists and coincides with the generalized directional derivative of $V(x)$ at x. Note a locally Lipschitz and convex function is regular. The following two lemmas are useful for proving the stability of discontinuous systems using nonsmooth Lyapunov functions. The next result can be found in Shevitz and Paden (1994), Bacciotti and Ceragioli (1999), Cortés and Bullo (2005).

Lemma 2.1: *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz and regular function. Suppose the initial state is x_0 and let $\Omega(x_0)$ be the connected component of $\{x \in \mathbb{R}^n \mid V(x) \leq V(x_0)\}$ containing x_0. Assume the set $\Omega(x_0)$ is bounded. If $\max \widetilde{\mathcal{L}}_f V(x) \leq 0$ or $\widetilde{\mathcal{L}}_f V(x) = \emptyset$ for all $x \in \Omega(x_0)$, then $\Omega(x_0)$ is strongly invariant for (2). Let*

$$Z_{f,V} = \{x \in \mathbb{R}^n \mid 0 \in \widetilde{\mathcal{L}}_f V(x)\}. \tag{6}$$

Then any solution of (2) starting from x_0 converges to the largest weakly invariant set M contained in $\overline{Z}_{f,V} \cap \Omega(x_0)$. Furthermore, if the set M is a finite collection of points, then the limit of all solutions starting from x_0 exists and equals one of them.

The next result can be found in Paden and Sastry (1987), Cortés and Bullo (2005).

Lemma 2.2: *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz and regular function. Suppose the initial state is x_0 and let S be a compact and strongly invariant set for (2). If $\max \widetilde{\mathcal{L}}_f V(x) \leq -\kappa < 0$ almost everywhere on $S \setminus Z_{f,V}$, then any solution of (2) starting at $x_0 \in S$ reaches $Z_{f,V} \cap S$ in finite time. The convergence time is upper bounded by $\left(V(x_0) - \min_{x \in S} V(x)\right)/\kappa$.*

3 Problem Formulation

In this section, we first describe the formation control problem that we are going to solve. Then we propose a distributed bearing-based control law and derive the closed-loop system dynamics.

3.1 *Control Objective*

Consider n ($n \geq 3$) vehicles in the plane. Denote the position of vehicle i as $z_i \in \mathbb{R}^2$, $i \in \{1, \dots, n\}$. The dynamics of each vehicle is modeled as

$$\dot{z}_i = u_i,$$

where $u_i \in \mathbb{R}^2$ is the control input to be designed. The target formation considered in this paper is an angle-constrained cyclic formation. The underlying information flow among the vehicles is

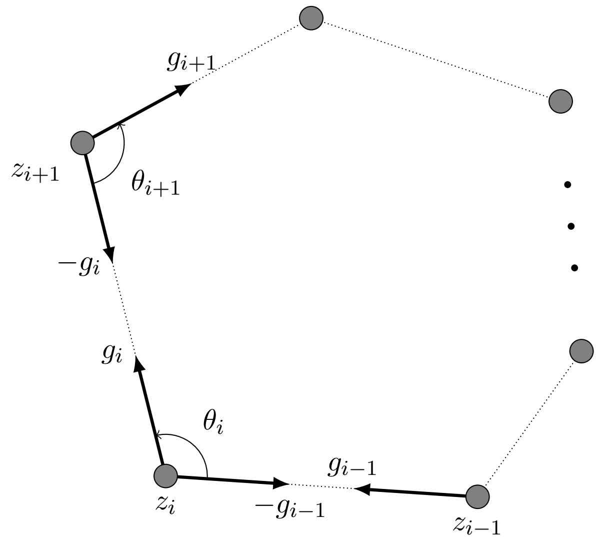


Figure 1.: An illustration of cyclic formations.

described by an undirected cycle graph with fixed topology. By indexing the vehicles properly, we can have $\mathcal{N}_i = \{i-1, i+1\}$ for $i \in \{1, \dots, n\}$, which means vehicles $i-1$ and $i+1$ are the neighbors of vehicle i. Then vehicle i can obtain the bearings of vehicles $i-1$ and $i+1$ by using, for example, a monocular camera. Note that the indices $i-1$ and $i+1$ are taken modulo n in this paper. Let $e_i \triangleq z_{i+1} - z_i$. Then the unit-length vector

$$g_i \triangleq \frac{e_i}{\|e_i\|}$$

characterizes the relative bearing between vehicles i and $i+1$ (see Figure 1). Thus the bearing measurements obtained by vehicle i consist of $-g_{i-1}$ and g_i, which are the relative bearings of vehicle $i-1$ and vehicle $i+1$, respectively.

Denote $\theta_i \in [0, 2\pi)$ as the angle subtended by vehicles $i-1$ and $i+1$ at vehicle i. The angle θ_i is specifically defined in the following way: rotating $-g_{i-1}$ counterclockwise through an angle θ_i about vehicle i yields g_i (see Figure 1). That can be mathematically expressed as

$$g_i = R(\theta_i)(-g_{i-1}), \tag{7}$$

where $R(\cdot)$ is the rotation matrix given in (1). By defining θ_i in (7), the angles θ_i and θ_{i+1} are on the same side of edge e_i for all $i \in \{1, \dots, n\}$. As a result, the quantity $\sum_{i=1}^n \theta_i$ is invariant to the positions of the vehicles because the sum of the interior angles of a polygon is constant. Hence $\sum_{i=1}^n \theta_i(t) \equiv \sum_{i=1}^n \theta_i(0)$ for all $t \in [0, +\infty)$. The angle θ_i is specified as a constant value $\theta_i^* \in [0, 2\pi)$ in the target formation. The target angles $\{\theta_i^*\}_{i=1}^n$ should be feasible such that there exist formations satisfying the angles constraints.

The problem we investigate in this paper is formally stated as below.

Assumption 3.1: In the initial formation, no two vehicles coincide with each other, i.e., $z_i(0) \neq z_j(0)$ for all $i \neq j$.

Assumption 3.2: In the target formation, $\theta_i^* \neq 0$ and $\theta_i^* \neq \pi$ for all $i \in \{1, \dots, n\}$.

Remark 1: Assumption 3.2 means no three consecutive vehicles in the target formation are collinear. The collinear case is a theoretical difficulty in many formation control problems (see, for example, (Krick et al. 2009, Dörfler and Francis 2010, Huang et al.)). In practice, bearings are usually measured by optical sensors such as cameras. Hence vehicle i would not be able to measure the bearings of its two neighbors simultaneously when $\theta_i = 0$ due to line-of-sight occlusion. On the other hand, the field-of-view of a monocular camera usually is less than 180 degree. Hence vehicle i would also not able to measure the bearings of its two neighbors simultaneously when $\theta_i = \pi$ due to limited field-of-view. Thus Assumption 3.2 is reasonable from the practical point of view.

Problem 3.3: *Under Assumptions 3.1 and 3.2, design control input u_i for vehicle i, $i \in \{1, \dots, n\}$, based only on the bearing measurements $\{-g_{i-1}, g_i\}$ such that θ_i converges to θ_i^* in finite time. During the formation evolution, collision avoidance between any vehicles (no matter they are neighbors or not) should be guaranteed.*

3.2 *Proposed Control Law*

We next propose a control law to solve Problem 3.3. Define the *angle error* for vehicle i as

$$\varepsilon_i = \cos\theta_i - \cos\theta_i^* = -g_i^{\mathrm{T}} g_{i-1} - \cos\theta_i^*, \tag{8}$$

where the second equality is due to $-g_i^{\mathrm{T}} g_{i-1} = \|g_i\|\|g_{i-1}\| \cos\theta_i = \cos\theta_i$. The proposed control law for vehicle i is

$$u_i = \mathrm{sgn}(\varepsilon_i)(g_i - g_{i-1}), \tag{9}$$

where

$$\mathrm{sgn}(\varepsilon_i) = \begin{cases} 1 & \text{if } \varepsilon_i > 0 \\ 0 & \text{if } \varepsilon_i = 0 \\ -1 & \text{if } \varepsilon_i < 0 \end{cases}.$$

For vector arguments, $\mathrm{sgn}(\cdot)$ is defined component-wise. Inspired by the control law in (Basiri et al. 2010), the proposed control law (9) has a clear geometric meaning: the control input vector u_i is along the bisector of the angle θ_i.

3.3 *Error Dynamics*

We next derive the error dynamics of the closed-loop system under control law (9).

Denote $\varepsilon = [\varepsilon_1, ..., \varepsilon_n]^{\mathrm{T}} \in \mathbb{R}^n$. Recall g_i is defined as $g_i = e_i/\|e_i\|$. Then the time derivative of g_i is

$$\dot{g}_i = \frac{\dot{e}_i}{\|e_i\|} - \frac{e_i}{\|e_i\|^2}\frac{\mathrm{d}\|e_i\|}{\mathrm{d}t} = \frac{\dot{e}_i}{\|e_i\|} - \frac{e_i}{\|e_i\|^2}\frac{e_i^{\mathrm{T}}}{\|e_i\|}\dot{e}_i = \frac{1}{\|e_i\|}\left(I - \frac{e_i}{\|e_i\|}\frac{e_i^{\mathrm{T}}}{\|e_i\|}\right)\dot{e}_i = \frac{1}{\|e_i\|}P_i\dot{e}_i, \tag{10}$$

where $P_i = I - g_i g_i^{\mathrm{T}}$. Matrix P_i plays an important role in the stability analysis in this paper. Geometrically P_i is an orthogonal projection matrix which can orthogonally project any vector onto the orthogonal compliment of g_i. The algebraic properties of P_i are listed below.

Lemma 3.4: *Matrix P_i satisfies:*

(i) $P_i^{\mathrm{T}} = P_i$ *and* $P_i^2 = P_i$.
(ii) P_i *is positive semi-definite.*
(iii) $\mathrm{Null}(P_i) = \mathrm{span}\{g_i\}$.

Proof

(i) The two properties are trivial to check.
(ii) For any $x \in \mathbb{R}^2$, since $P_i^2 = P_i$ and $P_i^{\mathrm{T}} = P_i$, we have $x^{\mathrm{T}} P_i x = x^{\mathrm{T}} P_i^{\mathrm{T}} P_i x = \|P_i x\|^2 \geq 0$.

(iii) First, it is easy to see $P_i g_i = 0$ and hence $g_i \in \text{Null}(P_i)$. Second, for any $x \in \mathbb{R}^2$, we have $P_i x = x - (g_i^{\mathrm{T}} x) g_i$. Clearly $P_i x = 0$ only if x is parallel to g_i. Thus $\text{Null}(P_i) = \text{span}\{g_i\}$.

□

Based on (10), we obtain the dynamics of ε as below.

Theorem 3.5 : *The ε-dynamics under control law* (9) *is*

$$\dot{\varepsilon} = -A\,\text{sgn}(\varepsilon), \tag{11}$$

where $A \in \mathbb{R}^{n\times n}$ and all of the entries of A are zero except

$$\begin{aligned} [A]_{i(i-1)} &= \frac{1}{\|e_{i-1}\|} g_i^{\mathrm{T}} P_{i-1} g_{i-2}, \\ [A]_{ii} &= \frac{1}{\|e_{i-1}\|} g_i^{\mathrm{T}} P_{i-1} g_i + \frac{1}{\|e_i\|} g_{i-1}^{\mathrm{T}} P_i g_{i-1}, \\ [A]_{i(i+1)} &= \frac{1}{\|e_i\|} g_{i-1}^{\mathrm{T}} P_i g_{i+1}, \end{aligned} \tag{12}$$

for all $i \in \{1, \ldots, n\}$.

Proof Recall $e_i = z_{i+1} - z_i$. Then substituting control law (9) into $\dot{e}_i = \dot{z}_{i+1} - \dot{z}_i$ yields

$$\begin{aligned} \dot{e}_i &= \dot{z}_{i+1} - \dot{z}_i \\ &= \text{sgn}(\varepsilon_{i+1})(g_{i+1} - g_i) - \text{sgn}(\varepsilon_i)(g_i - g_{i-1}) \\ &= \text{sgn}(\varepsilon_{i+1}) g_{i+1} + \text{sgn}(\varepsilon_i) g_{i-1} - [\text{sgn}(\varepsilon_{i+1}) + \text{sgn}(\varepsilon_i)]\, g_i. \end{aligned} \tag{13}$$

Substituting (13) into (10) and using the fact that $P_i g_i = 0$ gives

$$\dot{g}_i = \frac{1}{\|e_i\|} P_i \left[\text{sgn}(\varepsilon_{i+1}) g_{i+1} + \text{sgn}(\varepsilon_i) g_{i-1}\right].$$

Recall $\varepsilon_i = -g_i^{\mathrm{T}} g_{i-1} - \cos\theta_i^*$ as defined in (8) and θ_i^* is constant. Then by the above equation we have

$$\begin{aligned} \dot{\varepsilon}_i &= -g_i^{\mathrm{T}} \dot{g}_{i-1} - g_{i-1}^{\mathrm{T}} \dot{g}_i \\ &= -\frac{1}{\|e_{i-1}\|} g_i^{\mathrm{T}} P_{i-1} \left[\text{sgn}(\varepsilon_i) g_i + \text{sgn}(\varepsilon_{i-1}) g_{i-2}\right] - \frac{1}{\|e_i\|} g_{i-1}^{\mathrm{T}} P_i \left[\text{sgn}(\varepsilon_{i+1}) g_{i+1} + \text{sgn}(\varepsilon_i) g_{i-1}\right] \\ &= -[A]_{i(i-1)}\, \text{sgn}(\varepsilon_{i-1}) - [A]_{ii}\, \text{sgn}(\varepsilon_i) - [A]_{i(i+1)}\, \text{sgn}(\varepsilon_{i+1}), \end{aligned} \tag{14}$$

where $[A]_{i(i-1)}$, $[A]_{ii}$ and $[A]_{i(i+1)}$ are given in (12). It is straightforward to see the matrix form of (14) is (11). □

We next prove the matrix A in (11) is symmetric positive semi-definite.

Corollary 3.6: *The matrix A in* (11) *is symmetric positive semi-definite. For any $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$,*

$$x^{\mathrm{T}} A x = \sum_{i=1}^{n} \frac{1}{\|e_i\|} \left(g_{i+1} x_{i+1} + g_{i-1} x_i\right)^{\mathrm{T}} P_i \left(g_{i+1} x_{i+1} + g_{i-1} x_i\right) \geq 0. \tag{15}$$

Proof In order to prove A is symmetric, we only need to prove $[A]_{(i+1)i} = [A]_{i(i+1)}$ for all i. By changing the index i in $[A]_{i(i-1)}$ in (12) to $i+1$, we obtain

$$[A]_{(i+1)i} = \frac{1}{\|e_i\|} g_{i+1}^{\mathrm{T}} P_i g_{i-1}.$$

It is clear that $[A]_{(i+1)i} = [A]_{i(i+1)}$ due to the symmetry of P_i. For any vector $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$, we have

$$\begin{aligned}
x^{\mathrm{T}} A x &= \sum_{i=1}^{n} [A]_{i(i-1)} x_i x_{i-1} + [A]_{ii} x_i^2 + [A]_{i(i+1)} x_i x_{i+1} \\
&= \sum_{i=1}^{n} \left(\frac{1}{\|e_{i-1}\|} g_i^{\mathrm{T}} P_{i-1} g_{i-2} \right) x_i x_{i-1} + \sum_{i=1}^{n} \left(\frac{1}{\|e_{i-1}\|} g_i^{\mathrm{T}} P_{i-1} g_i \right) x_i^2 \\
&\quad + \sum_{i=1}^{n} \left(\frac{1}{\|e_i\|} g_{i-1}^{\mathrm{T}} P_i g_{i-1} \right) x_i^2 + \sum_{i=1}^{n} \left(\frac{1}{\|e_i\|} g_{i-1}^{\mathrm{T}} P_i g_{i+1} \right) x_i x_{i+1} \\
&= \sum_{i=1}^{n} \left(\frac{1}{\|e_i\|} g_{i+1}^{\mathrm{T}} P_i g_{i-1} \right) x_{i+1} x_i + \sum_{i=1}^{n} \left(\frac{1}{\|e_i\|} g_{i+1}^{\mathrm{T}} P_i g_{i+1} \right) x_{i+1}^2 \\
&\quad + \sum_{i=1}^{n} \left(\frac{1}{\|e_i\|} g_{i-1}^{\mathrm{T}} P_i g_{i-1} \right) x_i^2 + \sum_{i=1}^{n} \left(\frac{1}{\|e_i\|} g_{i-1}^{\mathrm{T}} P_i g_{i+1} \right) x_i x_{i+1} \\
&= \sum_{i=1}^{n} \frac{1}{\|e_i\|} \left(g_{i+1} x_{i+1} + g_{i-1} x_i \right)^{\mathrm{T}} P_i \left(g_{i+1} x_{i+1} + g_{i-1} x_i \right) \geq 0,
\end{aligned}$$

where the last inequality is due to the fact that P_i is positive semi-definite. □

4 Formation Stability Analysis

In this section we analyze the stability of the error dynamics (11). By employing a locally Lipschitz Lyapunov function and the nonsmooth analysis tools introduced in Section 2.3, we prove that the origin $\varepsilon = 0$ is locally finite-time stable with collision avoidance guaranteed. In addition to the dynamics of ε, we also analyze the behaviors of the vehicle positions during formation evolution.

We first consider the problem of collision avoidance. On one hand, Assumption 3.1 states that no vehicles coincide with each other in the initial formation, i.e., $z_i(0) \neq z_j(0)$ for any $i \neq j$. On the other hand, control law (9) implies that $\|\dot{z}_i\| \leq \|g_i - g_{i-1}\| \leq 2$, which means that the maximum speed of each vehicle is two. Therefore, any two vehicles are not able to collide with each other (no matter they are neighbors or not) for all $t \in [0, T^*)$ where

$$T^* \triangleq \frac{\min_{i \neq j} \|z_i(0) - z_j(0)\|}{4}.$$

In the rest of the paper, we will only consider $t \in [0, T]$ with $T < T^*$. We will prove that the system can be stabilized within the finite time interval $[0, T]$.

Consider the Lyapunov function

$$V(\varepsilon)=\sum_{i=1}^{n}|\varepsilon_i|, \tag{16}$$

which is positive definite in ε. Note $V(\varepsilon)$ is locally Lipschitz and convex. Hence $V(\varepsilon)$ is also regular.

Theorem 4.1 : *For the error dynamics* (11) *and Lyapunov function* (16)*, the Filippov differential inclusion is $\dot{\varepsilon}\in -A\partial V(\varepsilon)$. The set-valued Lie derivative is given by*

$$\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon)=\{\ell\in\mathbb{R}\mid \exists\eta\in\partial V(\varepsilon),\ \forall\zeta\in\partial V(\varepsilon),\ -\zeta^{\mathrm{T}}A\eta=\ell\}. \tag{17}$$

When $\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon)\neq\emptyset$, for any $\ell\in\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon)$, there exsits $\eta\in\partial V(\varepsilon)$ such that

$$\ell=-\eta^{\mathrm{T}}A\eta\leq 0. \tag{18}$$

Proof Step 1: calculate the generalized gradient. By definition we have the generalized gradient as

$$\begin{aligned}\partial V(\varepsilon)=\{\eta=[\eta_1,\dots,\eta_n]^{\mathrm{T}}\in\mathbb{R}^n\mid \eta_i=&\,\mathrm{sgn}(\varepsilon_i)\text{ if }\varepsilon_i\neq 0\text{ and}\\ &\eta_i\in[-1,1]\text{ if }\varepsilon_i=0\text{ for }i\in\{1,\dots,n\}\}.\end{aligned}$$

Because $|\eta_i|=|\mathrm{sgn}(\varepsilon_i)|=1$ if $\varepsilon_i\neq 0$, we have the obvious but important fact that

$$\|\eta\|\geq 1,\quad \forall\eta\in\partial V(\varepsilon),\ \forall\varepsilon\neq 0. \tag{19}$$

Additionally, if $\varepsilon_i\neq 0$, then $\mathrm{Ln}(\{\mathrm{sgn}(\varepsilon_i)\})=\{\mathrm{sgn}(\varepsilon_i)\}$; and if $\varepsilon_i=0$, then $\mathrm{Ln}([-1,1])=\{0\}=\{\mathrm{sgn}(0)\}$. Thus we have the following useful property

$$\mathrm{Ln}(\partial V(\varepsilon))=\{\mathrm{sgn}(\varepsilon)\}. \tag{20}$$

Step 2: calculate the Filippov differential inclusion. Since $\|e_i(t)\|\neq 0$ for all i and all $t\in[0,T]$, the matrix A in (9) is continuous. Then by (Paden and Sastry 1987, Theorem 1, 5)), the Filippov differential inclusion associated with the system (11) can be calculated as

$$\dot{\varepsilon}\in\mathcal{F}[-A\,\mathrm{sgn}(\varepsilon)]=-A\mathcal{F}[\mathrm{sgn}(\varepsilon)]. \tag{21}$$

Substituting (20) into (21) yields

$$\mathcal{F}[\mathrm{sgn}(\varepsilon)]=\mathcal{F}[\mathrm{Ln}(\partial V(\varepsilon))]=\partial V(\varepsilon),$$

where the last equality uses the fact (5). Thus the Filippov differential inclusion in (21) can be rewritten as

$$\dot{\varepsilon}\in -A\partial V(\varepsilon). \tag{22}$$

Step 3: calculate the set-valued Lie derivative. The set-valued Lie derivative of $V(\varepsilon)$ with respect to (22) is given by

$$\begin{aligned}\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) &= \{\ell \in \mathbb{R} \mid \exists \xi \in -A\partial V(\varepsilon),\ \forall \zeta \in \partial V(\varepsilon),\ \zeta^{\mathrm{T}}\xi = \ell\} \\ &= \{\ell \in \mathbb{R} \mid \exists \eta \in \partial V(\varepsilon),\ \forall \zeta \in \partial V(\varepsilon),\ -\zeta^{\mathrm{T}}A\eta = \ell\}.\end{aligned}$$

The set $\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon)$ could be empty. When $\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) \neq \emptyset$, for any $\ell \in \widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon)$, there exists $\eta \in \partial V$ such that $\ell = -\zeta^{\mathrm{T}}A\eta$ for all $\zeta \in \partial V$. In particular, by choosing $\zeta = \eta$ we have $\ell = -\eta^{\mathrm{T}}A\eta \leq 0$. Note $-\eta^{\mathrm{T}}A\eta \leq 0$ is due to the fact that A is a positive semi-definite matrix as shown in Lemma 3.6. Therefore, we have either $\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) = \emptyset$ or $\max \widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) \leq 0$. □

4.1 *Main Stability Result*

We first introduce a number of useful results and then prove the local finite-time formation stability.

Given an angle $\alpha \in \mathbb{R}$ and a vector $x \in \mathbb{R}^2$, the angle between x and $R(\alpha)x$ is α. Thus for all nonzero $x \in \mathbb{R}^2$, $x^{\mathrm{T}}R(\alpha)x > 0$ when $\alpha \in (-\pi/2, \pi/2) \pmod{2\pi}$; $x^{\mathrm{T}}R(\alpha)x = 0$ when $\alpha = \pm\pi/2 \pmod{2\pi}$; and $x^{\mathrm{T}}R(\alpha)x < 0$ when $\alpha \in (\pi/2, 3\pi/2) \pmod{2\pi}$.

Lemma 4.2: *Let $g_i^{\perp} \triangleq R(\pi/2)g_i$. The properties of $g_i^{\perp}$ are listed as below.*

(i) $\|g_i^{\perp}\| = 1$ *and* $(g_i^{\perp})^{\mathrm{T}}g_i = 0$.
(ii) $P_i = g_i^{\perp}(g_i^{\perp})^{\mathrm{T}}$.
(iii) *For* $i \neq j$, $(g_i^{\perp})^{\mathrm{T}}g_j = -(g_j^{\perp})^{\mathrm{T}}g_i$.
(iv) $(g_i^{\perp})^{\mathrm{T}}g_{i-1} = \sin\theta_i$, *which implies* $(g_i^{\perp})^{\mathrm{T}}g_{i-1} > 0$ *if* $\theta_i \in (0, \pi)$*; and* $(g_i^{\perp})^{\mathrm{T}}g_{i-1} < 0$ *if* $\theta_i \in (\pi, 2\pi)$.

Proof See Appendix. □

Lemma 4.3: *Let $\mathcal{U} \triangleq \{x \in \mathbb{R}^n :\ x \neq 0$ and nonzero entries of x do not have the same sign$\}$. Suppose $B \in \mathbb{R}^{n\times n}$ is a symmetric positive semi-definite matrix with $\lambda_1(B) = 0$ and $\lambda_2(B) > 0$. If $\mathbf{1} = [1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$ is an eigenvector associated with the zero eigenvalue of B, then*

$$\inf_{x\in\mathcal{U}} \frac{x^{\mathrm{T}}Bx}{x^{\mathrm{T}}x} = \frac{\lambda_2(B)}{n}.$$

Remark 2: By the definition of $\mathcal{U}$, any $x \in \mathcal{U}$ should at least contain one positive entry and one negative entry. If the nonzero entries of x are all positive or negative, then $x \notin \mathcal{U}$.

With the above preparation, we are ready to prove the formation stability based on Theorem 4.1. Note if $\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) = \emptyset$, we have $\max \widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) = -\infty$ (see Section 2.3.3). Hence we need only to focus on the case of $\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) \neq \emptyset$.

Theorem 4.4: *Consider the set-valued Lie derivative given in* (17)*. When $\widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon) \neq \emptyset$, for any $\ell \in \widetilde{\mathcal{L}}_{-A\partial V}V(\varepsilon)$, there exsits $\eta \in \partial V(\varepsilon)$ such that*

$$\ell \leq -\frac{1}{\sum_{i=1}^{n}\|e_i\|}\eta^{\mathrm{T}}D^{\mathrm{T}}E^{\mathrm{T}}ED\eta, \tag{23}$$

where

$$E=\begin{bmatrix} 1 & -1 & 0 & \dots & 0\\ 0 & 1 & -1 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -1 & 0 & \dots & 0 & 1\end{bmatrix}\in\mathbb{R}^{n\times n},\ D=\begin{bmatrix} (g_1^\perp)^{\mathrm{T}}g_n & 0 & 0 & \dots & 0\\ 0 & (g_2^\perp)^{\mathrm{T}}g_1 & 0 & \dots & 0\\ 0 & 0 & (g_3^\perp)^{\mathrm{T}}g_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & (g_n^\perp)^{\mathrm{T}}g_{n-1}\end{bmatrix}\in\mathbb{R}^{n\times n}. \tag{24}$$

Proof By (15), we can rewrite $\ell=-\eta^{\mathrm{T}}A\eta$ in (18) as

$$\begin{aligned}\ell &= -\sum_{i=1}^n\frac{1}{\|e_i\|}\left(g_{i+1}\eta_{i+1}+g_{i-1}\eta_i\right)^{\mathrm{T}}P_i\left(g_{i+1}\eta_{i+1}+g_{i-1}\eta_i\right)\\ &\le -\frac{1}{\sum_{i=1}^n\|e_i\|}\sum_{i=1}^n\left(g_{i+1}\eta_{i+1}+g_{i-1}\eta_i\right)^{\mathrm{T}}P_i\left(g_{i+1}\eta_{i+1}+g_{i-1}\eta_i\right)\\ &= -\frac{1}{\sum_{i=1}^n\|e_i\|}\sum_{i=1}^n\left[\left(g_{i+1}\eta_{i+1}+g_{i-1}\eta_i\right)^{\mathrm{T}}g_i^\perp\right]^2 \quad \text{(By Lemma 4.2(ii))}\\ &= -\frac{1}{\sum_{i=1}^n\|e_i\|}\sum_{i=1}^n\left[(g_i^\perp)^{\mathrm{T}}g_{i+1}\eta_{i+1}+(g_i^\perp)^{\mathrm{T}}g_{i-1}\eta_i\right]^2\\ &= -\frac{1}{\sum_{i=1}^n\|e_i\|}h^{\mathrm{T}}h, \end{aligned}\tag{25}$$

where

$$\begin{aligned} h &= \begin{bmatrix}(g_1^\perp)^{\mathrm{T}}g_2\eta_2+(g_1^\perp)^{\mathrm{T}}g_n\eta_1\\ \vdots\\ (g_n^\perp)^{\mathrm{T}}g_1\eta_1+(g_n^\perp)^{\mathrm{T}}g_{n-1}\eta_n\end{bmatrix}\\ &= \begin{bmatrix}(g_1^\perp)^{\mathrm{T}}g_n & (g_1^\perp)^{\mathrm{T}}g_2 & 0 & \dots & 0\\ 0 & (g_2^\perp)^{\mathrm{T}}g_1 & (g_2^\perp)^{\mathrm{T}}g_3 & \dots & 0\\ 0 & 0 & (g_3^\perp)^{\mathrm{T}}g_2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ (g_n^\perp)^{\mathrm{T}}g_1 & 0 & \dots & 0 & (g_n^\perp)^{\mathrm{T}}g_{n-1}\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \vdots\\ \eta_n\end{bmatrix}\\ &= ED\eta \end{aligned}\tag{26}$$

with E and D given in (24). The last equality of (26) uses the fact that $(g_i^\perp)^{\mathrm{T}}g_{i-1}=-(g_{i-1}^\perp)^{\mathrm{T}}g_i$ as shown in Lemma 4.2(iii). Substituting (26) into (25) gives (23). □

Note that D is a diagonal matrix and E actually is an incidence matrix of a directed and connected cycle graph. We now present the main stability result.

Theorem 4.5 : *Under Assumptions 3.1 and 3.2, the equilibrium* $\varepsilon=0$ *of system* (11) *is locally finite-time stable. Collision avoidance between any vehicles (no matter they are neighbors or not) can be locally guaranteed.*

Proof Consider the time interval $[0,T]$ with $T<T^*$. Then $\|e_i(t)\|\neq 0$ and $\|e_i(t)\|\neq+\infty$ for all $t\in[0,T]$. We will prove that ε can converge to zero in the finite time interval $[0,T]$ if $\varepsilon(0)$ is sufficiently small.

Let $\Omega(\varepsilon(0)) \triangleq \{\varepsilon \in \mathbb{R}^n \mid V(\varepsilon) \leq V(\varepsilon(0))\}$. Since $V(\varepsilon) = \sum_{i=1}^n |\varepsilon_i| = \|\varepsilon\|_1$, the level set $\Omega(\varepsilon(0))$ is connected and compact. Because $\widetilde{\mathcal{L}}_{-A\partial V} V(\varepsilon) = \emptyset$ or $\max \widetilde{\mathcal{L}}_{-A\partial V} V(\varepsilon) \leq 0$ for any $\varepsilon \in \Omega(\varepsilon(0))$ as proved in Theorem 4.1, we have that $\Omega(\varepsilon(0))$ is strongly invariant to (11) over $[0, T]$ by Lemma 2.1.

Step 1: prove the nonzero entries of $D\eta$ do not have the same sign.

Denote $\delta_i = \theta_i - \theta_i^*$ and $\delta = [\delta_1, \ldots, \delta_n]^{\mathrm{T}} \in \mathbb{R}^n$. Consider the case of $\varepsilon \neq 0$ and hence $\delta \neq 0$. Because $\sum_{i=1}^n \theta_i \equiv \sum_{i=1}^n \theta_i^*$, we have $\sum_{i=1}^n \delta_i \equiv 0$. Thus the nonzero entries of δ *do not have the same sign* if $\delta \neq 0$. Let

$$w_i \triangleq \frac{\cos\theta_i - \cos\theta_i^*}{\theta_i - \theta_i^*}.$$

Then $\varepsilon_i = w_i \delta_i$ and hence

$$\varepsilon = W\delta,$$

where $W = \mathrm{diag}\{w_1, \ldots, w_n\} \in \mathbb{R}^{n\times n}$. Since $\lim_{\theta_i \to \theta_i^*} w_i = -\sin\theta_i^*$ by L'Hôpital's rule, the equations $\varepsilon_i = w_i\delta_i$ and $\varepsilon = W\delta$ are always valid even when $\theta_i - \theta_i^* = 0$.

Suppose $V(\varepsilon(0))$ is sufficiently small such that $\theta_i(0)$ is sufficiently close to θ_i^* and hence $\theta_i, \theta_i^* \in (0, \pi)$ or $\theta_i, \theta_i^* \in (\pi, 2\pi)$ for all $\varepsilon \in \Omega(\varepsilon(0))$. Then it is easy to see that $w_i < 0$ if $\theta_i, \theta_i^* \in (0, \pi)$, and $w_i > 0$ if $\theta_i, \theta_i^* \in (\pi, 2\pi)$. On the other hand, recall $(g_i^\perp)^{\mathrm{T}} g_{i-1} > 0$ when $\theta_i \in (0, \pi)$, and $(g_i^\perp)^{\mathrm{T}} g_{i-1} < 0$ when $\theta_i \in (\pi, 2\pi)$ as shown in Lemma 4.2(iv). Thus we have

$$(g_i^\perp)^{\mathrm{T}} g_{i-1} w_i < 0$$

for all $i \in \{1, \ldots, n\}$. Since $[D]_{ii} = (g_i^\perp)^{\mathrm{T}} g_{i-1}$, the above inequality implies that the diagonal entries of DW *have the same sign.* Thus as the nonzero entries in δ do not have the same sign, the nonzero entries of $DW\delta = D\varepsilon$ *do not have the same sign* either. Furthermore, because $\eta_i = \mathrm{sgn}(\varepsilon_i)$ if $\varepsilon_i \neq 0$, the nonzero entry ε_i has the same sign with η_i. As a result, the nonzero entries of $D\eta$ *do not have the same sign.* Thus we have $D\eta \in \mathcal{U}$ with $\mathcal{U}$ defined in Lemma 4.3.

Step 2: determine the negative upper bound of ℓ.

Note E is an incidence matrix of a directed and connected cycle graph. By (Godsil and Royle 2001, Theorem 8.3.1), we have $\mathrm{rank}(E) = n - 1$ and $\mathrm{Null}(E^{\mathrm{T}}E) = \mathrm{Null}(E) = \mathrm{span}\{\mathbf{1}\}$. Thus inequality (23) implies

$$\begin{aligned}
\ell &\leq -\frac{1}{\sum_{i=1}^n \|e_i\|} \frac{\lambda_2(E^{\mathrm{T}}E)}{n} \|D\eta\|^2 \quad \text{(By Lemma 4.3)} \\
&\leq -\frac{1}{\sum_{i=1}^n \|e_i\|} \frac{\lambda_2(E^{\mathrm{T}}E)}{n} \lambda_1(D^2) \|\eta\|^2 \\
&\leq -\frac{1}{\sum_{i=1}^n \|e_i\|} \frac{\lambda_2(E^{\mathrm{T}}E)}{n} \lambda_1(D^2), \qquad (27)
\end{aligned}$$

where the last inequality uses the fact $\|\eta\| \geq 1$ if $\varepsilon \neq 0$ as shown in (19).

Now we analyze the two terms, $\sum_{i=1}^n \|e_i\|$ and $\lambda_1(D^2)$, in (27). (i) Over the finite time interval $[0, T]$, the quantity $\sum_{i=1}^n \|e_i\|$ cannot go to infinity because the vehicle speed is finite. Hence there exists a constant $\gamma > 0$ such that $\sum_{i=1}^n \|e_i\| \leq \gamma$. (ii) Since D is diagonal, we have $\lambda_1(D^2) = \min_i [D]_{ii}^2$. At the equilibrium point $\varepsilon = 0$ (i.e., $\theta_i = \theta_i^*$ for all i), we have $[D]_{ii} = (g_i^\perp)^{\mathrm{T}} g_{i-1} \neq 0$ because $\theta_i^* \neq 0$ or π as stated in Assumption 3.2. By continuity, we can still have $[D]_{ii} \neq 0$ for all $\varepsilon \in \Omega(\varepsilon(0))$ if $\varepsilon(0)$ is sufficiently small. Because $\Omega(\varepsilon(0))$ is compact, there exists a lower bound

β such that $\lambda_1(D^2) \geq \beta$ for all $\varepsilon \in \Omega(\varepsilon(0))$. By (i) and (ii), inequality (27) can be rewritten as

$$\ell \leq -\frac{\beta\lambda_2(E^{\mathrm{T}}E)}{\gamma n} \triangleq -\kappa < 0, \quad \forall \varepsilon \in \Omega(\varepsilon(0)) \setminus \{0\}. \tag{28}$$

Step 3: draw the stability conclusion.

If $\varepsilon = 0$ we have $0 \in \widetilde{\mathcal{L}}_{-A\partial V} V(\varepsilon)$ because of (17) and the fact that $0 \in \partial V(0)$; if $\varepsilon \neq 0$ we have $0 \notin \widetilde{\mathcal{L}}_{-A\partial V} V(\varepsilon)$ because $\max \widetilde{\mathcal{L}}_{-A\partial V} V(\varepsilon) < 0$ by (28). Thus by the definition (6), we have

$$Z_{-A\,\mathrm{sgn}(\varepsilon), V(\varepsilon)} = \{0\}. \tag{29}$$

Based on (28), (29) and Lemma 2.2, any solution of (11) starting from $\varepsilon(0)$ converges to $\varepsilon = 0$ in finite-time, and the convergence time is upper bounded by $V(\varepsilon(0))/\kappa$. Thus if $V(\varepsilon(0))$ satisfies

$$\frac{V(\varepsilon(0))}{\kappa} < T < T^*, \tag{30}$$

then the system can be stabilized within the time interval $[0, T]$ during which collision avoidance between any vehicles can be guaranteed. □

While the local formation stability is proved in Theorem 4.5, the convergence region of the equilibrium $\varepsilon = 0$ is not given. We next give a sufficient condition on $\varepsilon(0)$ to guarantee the convergence and collision avoidance.

Corollary 4.6: *Let* $\Delta_i \triangleq \min\{\theta_i^*, |\theta_i^* - \pi|, 2\pi - \theta_i^*\}$ *where* $i \in \{1, \dots, n\}$. *There exists* ξ *such that* $0 < \xi < \min_i \Delta_i$. *Let* $\bar{\varepsilon}_i \triangleq \min\{|\cos(\theta_i^* + \xi) - \cos\theta_i^*|, |\cos(\theta_i^* - \xi) - \cos\theta_i^*|\}$, $\zeta \triangleq \min_i\{\theta_i^* - \xi, |\pi - \theta_i^*| - \xi, 2\pi - \theta_i^* - \xi\}$ *and* $\gamma \triangleq \sum_{i=1}^n \|e_i(0)\| + 4T$. *Under Assumptions 3.1 and 3.2, the proposed control law guarantees the convergence of* ε *to zero in* $[0, T]$ *with collision avoidance between any vehicles if*

$$V(\varepsilon(0)) < \min\left\{\min_i \bar{\varepsilon}_i, \frac{\sin^2\zeta\lambda_2(E^{\mathrm{T}}E)}{\gamma n} T\right\}. \tag{31}$$

Proof The proof of Theorem 4.5 requires $\varepsilon(0)$ to be sufficiently small such that the following three conditions hold: (i) $\lambda_1(D^2) = \min_i[D^2]_{ii} > 0$ for all $\varepsilon \in \Omega(\varepsilon(0))$; (ii) θ_i^* and $\theta(t)$ for all $t \in [0, T]$ are both in $(0, \pi)$ or $(\pi, 2\pi)$; (iii) $V(\varepsilon(0))/\kappa < T$. Note condition (iii) ensures the collision avoidance.

Step 1: analyze condition (i). Recall $[D]_{ii} = (g_i^\perp)^{\mathrm{T}} g_{i-1} = \sin\theta_i$ as proved in Lemma 4.2. Hence $\min_i[D^2]_{ii} > 0$ if $\theta_i(t) \neq 0$ and $\theta_i(t) \neq \pi$ for all $t \in [0, T]$. Thus condition (ii) implies condition (i).

Step 2: analyze condition (ii). Denote $\Delta_i \triangleq \min\{\theta_i^*, |\theta_i^* - \pi|, 2\pi - \theta_i^*\}$. There exists ξ such that $0 < \xi < \min_i \Delta_i$. Let $\bar{\varepsilon}_i \triangleq \min\{|\cos(\theta_i^* + \xi) - \cos\theta_i^*|, |\cos(\theta_i^* - \xi) - \cos\theta_i^*|\}$. Then we have the following sufficient condition: if $\varepsilon(0)$ satisfies

$$V(\varepsilon(0)) < \min_i \bar{\varepsilon}_i, \tag{32}$$

then condition (ii) holds. To see that, for any $j \in \{1, \dots, n\}$, we have $|\varepsilon_j(t)| \leq \sum_{i=1}^n |\varepsilon_i(t)| = V(\varepsilon(t)) \leq V(\varepsilon(0)) < \min_i \bar{\varepsilon}_i \leq \bar{\varepsilon}_j$. Thus $|\varepsilon_j(t)| < \bar{\varepsilon}_j$ for all $t \in [0, T]$. Since the cosine function is monotone in $(0, \pi)$ or $(\pi, 2\pi)$, we have $|\varepsilon_j(t)| < \bar{\varepsilon}_j \Longrightarrow |\theta_i(t) - \theta_i^*| < \xi$ and hence condition (ii) holds. It should be noted $\bar{\varepsilon}_i \neq 0$ and hence the set of $\varepsilon(0)$ that satisfies (32) is always nonempty.

Further define $\zeta \triangleq \min_i\{\theta_i^* - \xi, |\pi - \theta_i^*| - \xi, 2\pi - \theta_i^* - \xi\}$. Then $|\theta_i(t) - \theta_i^*| < \xi$ implies $\theta_i(t) > \zeta$, $|\pi - \theta_i(t)| > \zeta$ and $2\pi - \theta_i(t) > \zeta$ for all $t \in [0, T]$. Thus $[D^2]_{ii} = \sin^2\theta_i > \sin^2\zeta$. Hence we have $\beta = \sin^2\zeta$, where β is the lower bound of $\lambda_1(D^2)$ as defined in the proof of Theorem 4.5.

Step 3: analyze condition (iii). We first identify an upper bound of $\sum_{i=1}^{n}\|e_i\|$. Since the speed of each vehicle is bounded above by two. It is easy to see $\sum_{i=1}^{n}\|e_i(t)\| \le \sum_{i=1}^{n}\|e_i(0)\| + 4T \triangleq \gamma$ for all $t \in [0, T]$. Therefore, we have κ defined in (28) as

$$\kappa = \frac{\sin^2 \zeta \lambda_2(E^{\mathrm{T}}E)}{\gamma n},$$

substituting which into (30) yields

$$V(\varepsilon(0)) < \frac{\sin^2 \zeta \lambda_2(E^{\mathrm{T}}E)}{\gamma n} T.$$

Therefore, if $V(\varepsilon(0))$ satisfies (31), then the three conditions are satisfied. By Theorem 4.5, the convergence of ε and collision avoidance between any vehicles can be guaranteed. □

Up to this point, the stability of the ε-dynamics has been proved. From control law (9), it is trivial to see that $\dot{z}_i = 0$ if $\varepsilon_i = 0$. Hence each vehicle will converge to a finite stationary finial position within finite time. Additionally, suppose the target formation is achieved at time $t_f < V(\varepsilon(0))/\kappa$. Since $\|\dot{z}_i\| \le \|g_i - g_{i-1}\| \le 2$, we have $\|z_i(t_f) - z_i(0)\| \le 2t_f \le 2V(\varepsilon(0))/\kappa$. Therefore, the final converged position $z_i(t_f)$ will be very close to its initial position $z_i(0)$ if the initial angle error $\varepsilon(0)$ is sufficiently small. In other words, the final converged formation will not be far away from the initial formation given small initial angle errors.

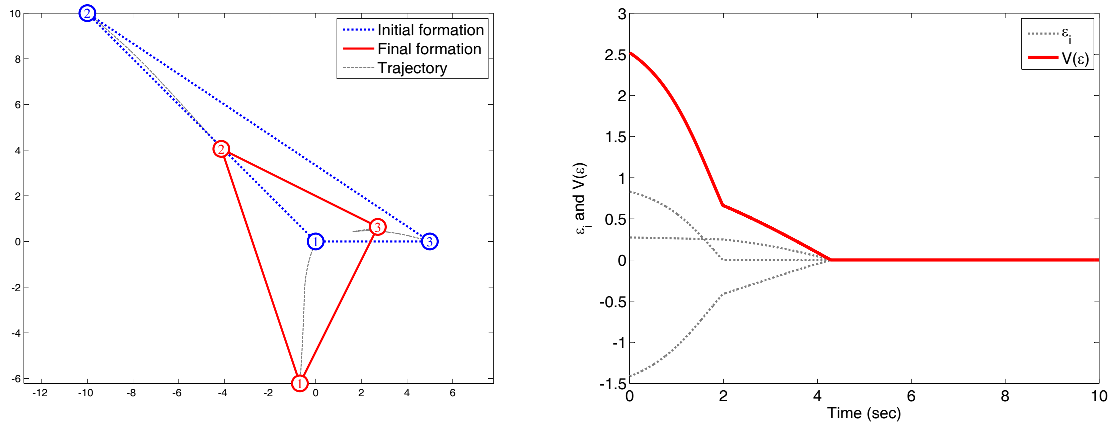
5 Simulation Results

In this section, we present simulation results to illustrate the preceding theoretical analysis. Figures 2, 3, 4 and 5 respectively show the formation control of three, four, five and eight vehicles. Note the five point star shown in figure 3 is not a normal polygon. But this kind of formations still have underlying graphs as cycles and hence can be stabilized by the proposed control law. As shown in the simulation, the proposed control law can efficiently reduce the angle errors and stabilize the formation in finite time. In our stability proof, we assume that the initial angle error $\varepsilon(0)$ should be sufficiently small such that $\theta_i(0)$ and θ_i^* are in either $(0, \pi)$ or $(\pi, 2\pi)$. However, as shown in figures 3 and 5, the formation can still be stabilized even if $\theta_i(0)$ and θ_i^* may be respectively in the two intervals $(0, \pi)$ and $(\pi, 2\pi)$. Hence the simulation suggests that the attractive region of the target formation by the proposed control law is not necessarily small.

6 Conclusions

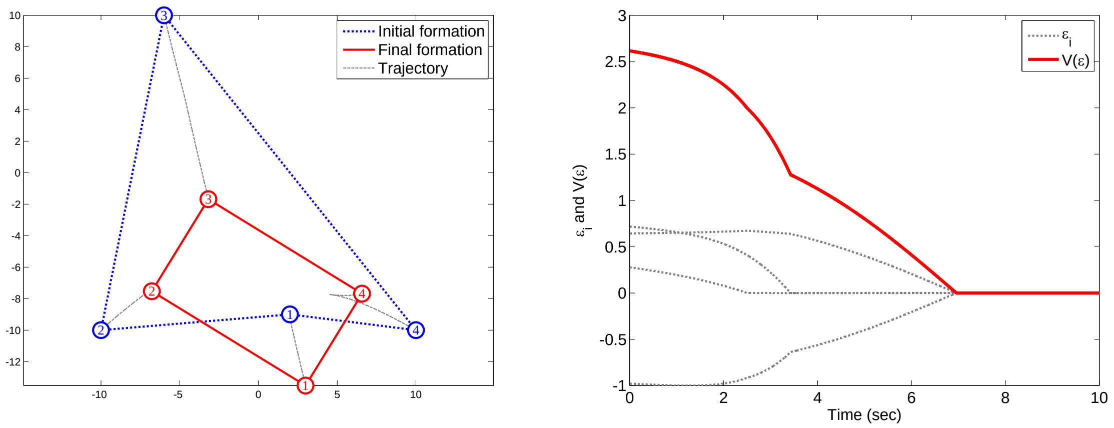
In this paper, we proposed a distributed control law to stabilize angle-constrained cyclic formations using bearing-only measurements. Compared to the existing work, the proposed control law can handle cyclic formations with an arbitrary number of vehicles, and only requires the sign information of the angle errors. By using nonsmooth stability analysis tools, we proved that the formation is locally finite-time stable with collision avoidance guaranteed.

Compared to the conventional position-based formation control, bearing-based formation control does possess a number of unique features. For example, the formation shape and the formation scale are uncontrollable by bearing-based formation control since the inter-vehicle distances are uncontrollable. But these limitations of bearing-based formation control can be all well overcomed in the future. For example, the formation shape can be specified by assigning a rigid underlying graph to a formation. To control the scale of a formation, we can introduce some leader vehicles whose inter-vehicle distances are controllable. Then given appropriate underlying



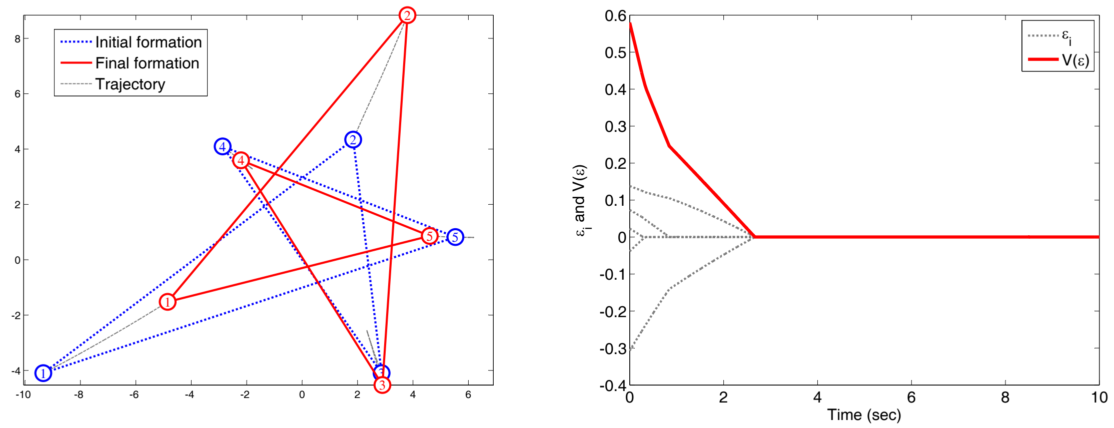
(a) Vehicle trajectory (b) Angle error and Lyapunov function

Figure 2.: Control results by the proposed control law with $n = 3$, $\theta_1^* = \theta_2^* = 45$ deg and $\theta_3^* = 90$ deg.



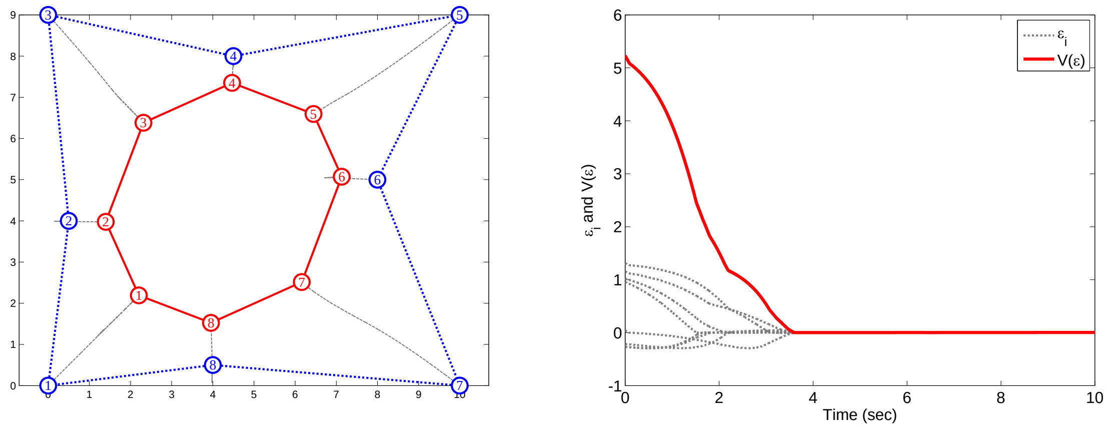
(a) Vehicle trajectory (b) Angle error and Lyapunov function

Figure 3.: Control results by the proposed control law with $n = 4$ and $\theta_1^* = \cdots = \theta_4^* = 90$ deg.



(a) Vehicle trajectory (b) Angle error and Lyapunov function

Figure 4.: Control results by the proposed control law with $n = 5$ and $\theta_1^* = \cdots = \theta_5^* = 36$ deg.



(a) Vehicle trajectory (b) Angle error and Lyapunov function

Figure 5.: Control results by the proposed control law with $n = 8$ and $\theta_1^* = \cdots = \theta_8^* = 135$ deg.

sensing graph, the formation scale can be controlled by tuning the distances among the leader vehicles. These topics are very interesting and challenging directions for future research.

Here are some other interesting topics for future research on bearing-based formation control. First, this paper only considers formations with cycle graphs. One of our immediate research plans is to extend the work in this paper to the cases with more complicated graphs. The extension will be non-trivial, but the structure of the stability analysis in this paper is believed to be useful for future research on more complicated cases. Second, we assume the underlying graph is undirected in this paper. It may be more practical for the underlying sensing graphs to be directed. Finally, the vehicle dynamics is assumed as a single integrator in this work. More complicated vehicle dynamics and system uncertainties need to be considered in the future.

Appendix

Proof [Proof of Lemma 4.2]

(i) The two equations are obvious.

(ii) Denote $G_i = [g_i, g_i^\perp] \in \mathbb{R}^{2\times 2}$. It is clear that G_i is an orthogonal matrix satisfying $G_i^{\mathrm{T}} G_i = G_i G_i^{\mathrm{T}} = I$. Hence we have

$$g_i g_i^{\mathrm{T}} + g_i^\perp (g_i^\perp)^{\mathrm{T}} = G_i G_i^{\mathrm{T}} = I.$$

Thus $g_i^\perp (g_i^\perp)^{\mathrm{T}} = I - g_i g_i^{\mathrm{T}} = P_i$.

(iii) $(g_i^\perp)^{\mathrm{T}} g_j = g_i^{\mathrm{T}} R^{\mathrm{T}}(\pi/2) g_j = g_i^{\mathrm{T}} R(-\pi/2) g_j = g_i^{\mathrm{T}} R(-\pi) R(\pi/2) g_j = g_i^{\mathrm{T}} R(-\pi) g_j^\perp = g_i^{\mathrm{T}}(-I) g_j^\perp = -(g_j^\perp)^{\mathrm{T}} g_i$.

(iv) By the definition of θ_i, we have $g_i = R(\theta_i)(-g_{i-1})$ and hence $g_{i-1} = -R(-\theta_i) g_i$. Then

$$\begin{aligned}
(g_i^\perp)^{\mathrm{T}} g_{i-1} &= -g_i^{\mathrm{T}} R\left(-\frac{\pi}{2}\right) R(-\theta_i) g_i \\
&= -g_i^{\mathrm{T}} R\left(-\frac{\pi}{2} - \theta_i\right) g_i \\
&= -\|g_i\| \left\| R\left(-\frac{\pi}{2} - \theta_i\right) g_i \right\| \cos\left(-\frac{\pi}{2} - \theta_i\right) \\
&= \sin\theta_i.
\end{aligned}$$

Then it is straightforward to see the rest results in Lemma 4.2(iv).

□

Proof [Proof of Lemma 4.3] By orthogonally projecting $x \in \mathcal{U}$ to $\mathbf{1}$ and the orthogonal complement of $\mathbf{1}$, we decompose x as

$$x = x_0 + x_1,$$

where $x_0 \in \mathrm{Null}(B)$ and $x_1 \perp \mathrm{Null}(B)$. Let φ be the angle between $\mathbf{1}$ and x. Then we have

$$\begin{aligned} x^{\mathrm{T}} B x &= x_1^{\mathrm{T}} B x_1 \\ &\geq \lambda_2(B) \|x_1\|^2 \\ &= \lambda_2(B) \sin^2 \varphi \|x\|^2. \end{aligned} \tag{33}$$

By the definition of $\mathcal{U}$, any x in $\mathcal{U}$ would not be in span$\{\mathbf{1}\}$. That means $\varphi \neq 0$ or π and hence $\sin\varphi \neq 0$. We next identify the positive infimum of $\sin\varphi$. The open set $\mathcal{U}$ is enclosed by the hyper-planes $[x]_i = 0$ with $i \in \{1, \ldots, n\}$. And $\mathbf{1}$ is isolated from any $x \in \mathcal{U}$ by the hyper-planes. Denote the boundary of $\mathcal{U}$ as $\partial\mathcal{U}$. Then we have $\inf_{x\in\mathcal{U}} \varphi = \min_{x\in\partial\mathcal{U}} \angle(x, \mathbf{1})$ and $\sup_{x\in\mathcal{U}} \varphi = \max_{x\in\partial\mathcal{U}} \angle(x, \mathbf{1})$. Denote $p_i \in \mathbb{R}^n$ as the orthogonal projection of $\mathbf{1}$ on the hyper-plane $[x]_i = 0$. Then $\min_{x\in\partial\mathcal{U}} \angle(x, \mathbf{1}) = \angle(p_i, \mathbf{1})$ and $\max_{x\in\partial\mathcal{U}} \angle(x, \mathbf{1}) = \angle(-p_i, \mathbf{1})$. Note the ith entry of p_i is zero and the others are one. It can be calculated that $\cos\angle(\pm p_i, \mathbf{1}) = \pm\sqrt{n-1}/\sqrt{n}$ and hence $\sin\angle(\pm p_i, \mathbf{1}) = 1/\sqrt{n}$. Thus

$$\inf_{x\in\mathcal{U}} \sin\varphi = \frac{1}{\sqrt{n}},$$

substituting which into (33) yields

$$\inf_{x\in\mathcal{U}} \frac{x^{\mathrm{T}} B x}{x^{\mathrm{T}} x} = \frac{\lambda_2(B)}{n}.$$

□

References

A. Bacciotti and F. Ceragioli. Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESAIM: Control, Optimisation and Calculus of Variations*, 4:361–376, January 1999.

M. Basiri, A. N. Bishop, and P. Jensfelt. Distributed control of triangular formations with angle-only constraints. *Systems & Control Letters*, 59:147–154, 2010.

S. P. Bhat and D. S. Bernstein. Finite-time stability of continuos autonomous systems. *SIAM Journal on Control and Optimization*, 38(3):751–766, 2000.

A. N. Bishop. Stabilization of rigid formations with direction-only constraints. In *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference*, pages 746–752, Orlando, FL, USA, December 2011.

M. Cao, C. Yu, and B. D. O. Anderson. Formation control using range-only measurements. *Automatica*, 47:776–781, 2011.

G. Chen, F. L. Lewis, and L. Xie. Finite-time distributed consensus via binary control protocols. *Automatica*, 47:1962–1968, 2011.

F. H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley, New York, 1983.

J. Cortés. Discontinuous dynamical systems. *IEEE Control Systems Magazine*, 28(3):36–73, 2008.

J. Cortés and F. Bullo. Coordination and geometric optimization via distributed dynamical systems. *SIAM Journal on Control and Optimization*, 44(5):1543–1574, 2005.

A. K. Das, R. Fierro, V. Kumar, J. P. Ostrowski, J. Spletzer, and C. J. Taylor. A vision-based formation control framework. *IEEE Transactions on Robotics and Automation*, 18(5):813–825, October 2002.

D. V. Dimarogonas and K. H. Johansson. Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control. *Automatica*, 46(4):695–700, April 2010.

F. Dörfler and B. Francis. Geometric analysis of the formation problem for autonomous robots. *IEEE Transactions on Automatic Control*, 55(10):2379–2384, October 2010.

T. Eren. Formation shape control based on bearing rigidity. *International Journal of Control*, 85(9):1361–1379, 2012.

A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers, 1988.

A. Franchi and P. R. Giordano. Decentralized control of parallel rigid formations with direction constraints and bearing measurements. In *Proceedings of the 51st IEEE Conference on Decision and Control*, pages 5310–5317, Hawaii, USA, December 2012.

C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, New York, 2001.

Y. Hong, Y. Xu, and J. Huang. Finite-time control for robot manipulators. *Systems & Control Letters*, 46:243–253, 2002.

Y. Hong, Z. P. Jiang, and G. Feng. Finite-time input-to-state stability and applications to finite-time control design. *SIAM Journal on Control and Optimization*, 48(7):4395–4418, 2010.

J. Hu, J. Xu, and L. Xie. Cooperative search and exploration in robotic networks. *Unmanned Systems*, 1(1):121–142, 2013.

H. Huang, C. Yu, and Q. Wu. Autonomous scale control of multiagent formations with only shape constraints. *International Journal of Robust and Nonlinear Control*. to appear.

J. Keller, D. Thakur, V. Dobrokhodov, K. Jones, M. Pivtoraiko, J. Gallier, I. Kaminer, and V. Kumar. A computationally efficient approach to trajectory management for coordinated aerial surveillance. *Unmanned Systems*, 1(1):59–74, 2013.

A. N. Kopeikin, S. S. Ponda, L. B. Johnson, and J. P. How. Dynamic mission planning for communication control in multiple unmanned aircraft teams. *Unmanned Systems*, 1(1):41–58, 2013.

L. Krick, M. E. Broucke, and B. A. Francis. Stabilization of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, 82(3):423–439, 2009.

Z. Lin, B. Francis, and M. Maggiore. Necessary and sufficient graphical conditions for formation control. *IEEE Transactions on Automatic Control*, 50(1):121–127, January 2005.

Y. Ma, S. Soatto, J. Kosecka, and S. Sastry. *An Invitation to 3D Vision*. Springer, New York, 2004.

G. L. Mariottini, F. Morbidi, D. Prattichizzo, N. V. Valk, N. Michael, G. Pappas, and K. Daniilidis. Vision-based localization for leader–follower formation control. *IEEE Transactions on Robotics*, 25(6):1431–1438, December 2009.

Z. Meng and Z. Lin. Distributed finite-time cooperative tracking of networked lagrange systems via local interactions. In *Proceedings of American Control Conference*, pages 4951–4956, Montreal, Canada, June 2012.

N. Moshtagh, N. Michael, A. Jadbabaie, and K. Daniilidis. Bearing-only control laws for balanced circular formations of ground robots. In *Proceedings of Robotics: Science and Systems*, Zurich, Switzerland, June 2008.

B. Paden and S. S. Sastry. A calculus for computing Filippovs differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transactions on Circuits and Systems*, 34(1):73–82, January 1987.

W. Ren and Y. Cao. *Distributed Coordination of Multi-agent Networks*. Springer-Verlag, London, 2011.

D. Shevitz and B. Paden. Lyapunov stability theory of nonsmooth systems. *IEEE Transactions on Automatic Control*, 39(9):1910–1914, September 1994.

Y.-P. Tian and Q. Wang. Global stabilization of rigid formations in the plane. *Automatica*, 49 (5):1436–1441, May 2013.

F. Xiao, L. Wang, J. Chen, and Y. Gao. Finite-time formation control for multi-agent systems. *Automatica*, 45:2605–2611, 2009.