International Journal of Control
Vol. 00, No. 00, Month 200x, 1–25

RESEARCH ARTICLE

Optimal Sensor Placement for Target Localization and Tracking in 2D and 3D

Shiyu Zhao[a], Ben M. Chen[b*] and Tong H. Lee[b]

[a] *Graduate School for Integrative Sciences and Engineering, National University of Singapore, Singapore*
[b] *Department of Electrical and Computer Engineering, National University of Singapore, Singapore*

(*March 14, 2013*)

This paper analytically characterizes optimal sensor placements for target localization and tracking in 2D and 3D. Three types of sensors are considered: bearing-only, range-only, and received-signal-strength. The optimal placement problems of the three sensor types are formulated as an identical parameter optimization problem and consequently analyzed in a unified framework. Recently developed frame theory is applied to the optimality analysis. We prove necessary and sufficient conditions for optimal placements in 2D and 3D. A number of important analytical properties of optimal placements are further explored. In order to verify the analytical analysis, we present a gradient control law that can numerically construct generic optimal placements.

Keywords: Fisher information matrix; gradient control; optimal sensor placement; target tracking; tight frame.

1 Introduction

Target localization and tracking using sensor networks has become an active research area in recent years. When localizing a target from noisy measurements of multiple sensors, the sensor placement can significantly affect the estimation accuracy. The term *sensor placement* as used here refers to the relative sensor-target geometry. This paper will address the *optimal* sensor placements that minimize the target estimation uncertainty. Optimal sensor placement problems are of not only theoretical interest but also significantly practical value.

In the literature, there are generally two kinds of mathematical formulations for optimal sensor placement problems. One is optimal control (Ousingsawat and Campbell 2007, Sinclair et al. 2008, Oshman and Davidson 1999, Miller and Rubinovich 2003) and the other is parameter optimization (Bishop et al. 2010, 2007, Bishop and Jensfelt 2009, Doğançay and Hmam 2008, Martínez and Bullo 2006, Zhang 1995, Doğançay 2007, Isaacs et al. 2009, Moreno-Salinas et al. 2011).

The optimal control formulations are usually adopted for cooperative path planning problems (Ousingsawat and Campbell 2007, Sinclair et al. 2008, Oshman and Davidson 1999), the aim of which is to estimate the target position on one hand and plan the path of sensor platforms to minimize the estimation uncertainty on the other hand. These problems are also referred as simultaneous localization and planning (SLAP) (Sinclair et al. 2008). In a SLAP problem, the target motion model and the sensor measurement model are respectively considered as the process and measurement models. Then the Kalman filter is usually applied to estimate the target position and to characterize the estimation variance. In order to minimize the dynamic

*Corresponding author. Email: bmchen@nus.edu.sg

ISSN: 0020-7179 print/ISSN 1366-5820 online

DOI: 10.1080/0020717YYxxxxxxxx
http://www.informaworld.com

estimation variance, an optimal control problem would be formulated. The disadvantage of this kind of formulation is that the optimal control with various constraints generally can only be solved by numerical methods. Analytical properties usually cannot be obtained.

Optimal sensor placement problems are also widely formulated as parameter optimization problems (Bishop et al. 2010, 2007, Bishop and Jensfelt 2009, Doğançay and Hmam 2008, Martínez and Bullo 2006, Zhang 1995, Doğançay 2007, Isaacs et al. 2009, Moreno-Salinas et al. 2011). The parameter optimization formulations assume that an initial estimate of the target position has already been obtained in other ways. Based on this initial estimate, the sensor positions are the parameters to be determined such that the target estimation uncertainty can be minimized. The target estimation uncertainty is usually characterized by the Fisher information matrix (FIM). The FIM is the inverse of the Cramer-Rao lower bound (CRLB), which is the minimum achievable estimation variance. An unbiased estimator that achieves the CRLB is called efficient. Hence an optimal placement, which optimizes a function (such as the determinant) of the FIM, can be interpreted as maximizing the target information gathered by the sensors or as minimizing the estimation variance of any efficient estimator.

In contrast to optimal control formulations, parameter optimization formulations can generally be solved analytically. The analytical solutions are important because they can provide valuable insights into the impact of sensor placements on target localization/tracking uncertainty. Motivated by this, we adopt the parameter optimization formulation to analyze optimal placements in this work. Our aim is to analytically determine the optimal sensor-target geometry based on an initial estimate of the target position. In practice, the initial estimate can be obtained by using, for example, Kalman filter. The optimal placement deployed based on the initial estimate is supposed to be able to improve the consequent target localization/tracking accuracy. It should be noted that we will not discuss target estimation or practical applications of optimal sensor placements in this paper. Interested readers may refer to (Martínez and Bullo 2006, Section 4) for a comprehensive example that illustrates how optimal sensor placements can be applied to cooperative target tracking.

Until now, most of the existing results have been only concerned with optimal sensor placements in 2D space (Bishop et al. 2010, 2007, Bishop and Jensfelt 2009, Doğançay and Hmam 2008, Martínez and Bullo 2006, Zhang 1995, Isaacs et al. 2009). Very few studies have tackled 3D cases (Moreno-Salinas et al. 2011). Analytical characterization of generic optimal sensor placements in 3D is still an open problem. In this paper we will extend the results in Bishop et al. (2010), Doğançay and Hmam (2008), Martínez and Bullo (2006), Bishop and Jensfelt (2009) from 2D to 3D. The extension is non-trivial. Maximizing the determinant of the FIM has been widely adopted as the criterion for optimal placements in 2D. This criterion, however, cannot be directly applied to 3D cases because the determinant of the FIM is hardly analytically tractable in 3D cases. Motivated by this, we propose a new criterion for optimal placements, which enables us to analytically characterize optimal placements in 2D and 3D. Our analysis based on the new criterion includes the existing results on 2D optimal placements as special cases.

The existing work on optimal sensor placement has addressed many sensor types such as bearing-only (Bishop et al. 2010, Doğançay and Hmam 2008, Zhao et al. 2012), range-only (Martínez and Bullo 2006, Bishop et al. 2010, Jourdan and Roy 2006), received-signal-strength (RSS) (Bishop and Jensfelt 2009), time-of-arrival (TOA) (Bishop et al. 2010, 2007), time-difference-of-arrival (TDOA) (Bishop et al. 2010, Isaacs et al. 2009), and Doppler (Bishop and Smith 2010). In this paper we consider three sensor types: bearing-only, range-only, and RSS. The three sensor types have been analyzed individually in the literature. In our work we present a *unified framework* for analyzing the optimal placements for the three sensor types. More specifically, we propose a new optimality criterion for optimal placements, based on which the objective functions for the three sensor types will be the same. Hence their optimal placements can be analyzed in a unified way. Moreover, it is notable that some researchers have studied optimal placements of *heterogeneous* sensor networks which contain different sensor types (Meng et al., Yang et al. 2011). Our work only considers *homogeneous* sensor networks that contain one single type of sensors. But we do not assume the sensor measurement qualities (characterized by noise variances) to be the same.

By employing recently developed frame theory, we will prove necessary and sufficient conditions for optimal placements in 2D and 3D. Frames provide a redundant and robust way for representing signals and are widely used in signal processing. One may refer to Kovačević and Chebira (2007a,b) for an introduction to frames. It might be interesting to ask why frames arise in optimal sensor placement problems. This question can be loosely answered from the point of view of redundancy. It is pointed out by Kovačević and Chebira (2007a): "Why and where would one use frames? The answer is obvious: anywhere where redundancy is a must." In our work, the redundancy can be interpreted as the ratio between the number of sensors and the space dimension. When the sensor number equals the space dimension, there is no redundancy in the system, then we can prove the necessary and sufficient condition of optimal placement without using frames. But when the sensor number is larger than the space dimension, our optimality analysis will heavily rely on frame theory.

The paper is organized as follows. Section 2 introduces preliminaries to frame theory. Section 3 presents a unified mathematical formulation for optimal placement problems of bearing-only, range-only, and RSS sensors in 2D and 3D. In Section 4, we present necessary and sufficient conditions for optimal placements. Section 5 further explores a number of important properties of optimal placements. In Section 6, a gradient control law is proposed to numerically verify our analytical analysis. Conclusions are drawn in Section 7.

2 Preliminaries to Frame Theory

Frames can be defined in any Hilbert space. Here we are only interested in d-dimensional Euclidean space $\mathbb{R}^d$ with $d \geq 2$. Let $\|\cdot\|$ be the Euclidean norm of a vector or the Frobenius norm of a matrix. As shown by Benedetto and Fickus (2003), Casazza et al. (2006), Kovačević and Chebira (2007a,b), a set of vectors $\{\varphi_i\}_{i=1}^n$ in $\mathbb{R}^d$ $(n \geq d)$ is called a frame if there exist constants $0 < a \leq b < +\infty$ so that for all $x \in \mathbb{R}^d$

$$a\|x\|^2 \leq \sum_{i=1}^{n} \langle x, \varphi_i \rangle^2 \leq b\|x\|^2, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors. The constants a and b are called the *frame bounds*. A frame $\{\varphi_i\}_{i=1}^n$ is called *unit-norm* if $\|\varphi_i\| = 1$ for all $i \in \{1, \dots, n\}$. Denote $\Phi = [\varphi_1, \dots, \varphi_n] \in \mathbb{R}^{d\times n}$. Because $\langle x, \varphi_i\rangle^2 = (x^T\varphi_i)^2 = x^T\varphi_i\varphi_i^T x$, inequality (1) can be rewritten as

$$a\|x\|^2 \leq x^T\Phi\Phi^T x \leq b\|x\|^2,$$

where the matrix $\Phi\Phi^T = \sum_{i=1}^n \varphi_i\varphi_i^T$ is called the *frame operator*. The frame bounds a and b obviously are the smallest and largest eigenvalues of $\Phi\Phi^T$, respectively. Since $a > 0$, $\Phi\Phi^T$ is positive definite and hence Φ is of full row rank. Therefore, the frame $\{\varphi_i\}_{i=1}^n$ spans $\mathbb{R}^d$. It is well known that d vectors in $\mathbb{R}^d$ form a basis if they span $\mathbb{R}^d$. Frame essentially is a generalization of the concept of basis. Unlike a basis, a frame have $n - d$ redundant vectors. The constant n/d is referred as the *redundancy* of the system. When $n/d = 1$, a frame would degenerate to a basis.

Tight frame is a particularly important concept in frame theory. A frame is *tight* when $a = b$. From (1) it is easy to see the frame $\{\varphi_i\}_{i=1}^n$ is tight when

$$\sum_{i=1}^{n} \varphi_i\varphi_i^T = aI_d. \tag{2}$$

Taking trace on both sides of (2) yields $a = \sum_{i=1}^{n} \|\varphi_i\|^2/d$. It is an important and fundamental problem in frame theory to construct a tight frame $\{\varphi_i\}_{i=1}^n$ that solves (2) with specified norms. This problem is also recognized as notoriously difficult (Casazza et al. 2012). One approach to this problem is to characterize tight frames as the minimizers of the *frame potential*

$$\mathrm{FP}\left(\{\varphi_i\}_{i=1}^n\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\varphi_i^T\varphi_j\right)^2. \tag{3}$$

Frame potential was first proposed by Benedetto and Fickus (2003) for unit-norm frames, and then generalized by Casazza et al. (2006) for frames with arbitrary norms.

We can find tight frames by minimizing the frame potential. The following concept of *irregularity* is crucial for characterizing the minimizers of the frame potential (Casazza et al. 2006, Kovačević and Chebira 2007a).

Definition 2.1 (Irregularity): For any positive non-increasing sequence $\{c_i\}_{i=1}^n$ with $c_1 \geq \cdots \geq c_n > 0$, and any integer d satisfying $1 \leq d \leq n$, denote k_0 as the smallest nonnegative integer k for which

$$c_{k+1}^2 \leq \frac{1}{d-k}\sum_{i=k+1}^{n} c_i^2. \tag{4}$$

The integer k_0 is called the irregularity of $\{c_i\}_{i=1}^n$ with respect to d.

Remark 1: The irregularity of a sequence is evaluated with respect to a particular positive integer d. The irregularity of a given sequence may be different when evaluated with respect to different positive integers. In this paper, we will omit mentioning this integer when the context is clear.

Because the index $k = d - 1$ always makes (4) hold, the irregularity k_0 always exists and satisfies

$$0 \leq k_0 \leq d-1.$$

When $k_0 = 0$, inequality (4) degenerates to the *fundamental inequality* (Casazza et al. 2006)

$$\max_{j=1,\ldots,n} c_j^2 \leq \frac{1}{d}\sum_{i=1}^{n} c_i^2. \tag{5}$$

In this paper we call the sequence $\{c_i\}_{i=1}^n$ *regular* when $k_0 = 0$, and *irregular* when $k_0 \neq 0$. The fundamental inequality (5) intuitively implies: a sequence is regular when no element is much larger than the others. Next we show several examples to illustrate the concept of irregularity.

Example 2.2 Consider a sequence $\{c_i\}_{i=1}^n$ with $c_1 = \cdots = c_n = c$ and any $d \leq n$. The fundamental inequality (5) holds because $1/d\sum_{i=1}^n c_i^2 = nc^2/d \geq c^2$. Thus $\{c_i\}_{i=1}^n$ is regular with respect to any integer $d \leq n$. This result will be frequently used in the sequel.

Example 2.3 Consider a sequence $\{c_i\}_{i=1}^4 = \{10, 1, 1, 1\}$ and $d = 3$. Note the feature of this sequence is that one element is much larger than the others. Because $10^2 > 1/3(10^2 + 1 + 1 + 1)$, the sequence is irregular with respect to $d = 3$. In order to determine the irregularity k_0, we need to further check if $\{c_i\}_{i=2}^4 = \{1, 1, 1\}$ is regular with respect to $d - 1 = 2$. Since the elements of $\{c_i\}_{i=2}^4$ equal to each other, $\{c_i\}_{i=2}^4$ is regular with respect to 2 as shown in Example 2.2. Hence the irregularity of $\{c_i\}_{i=1}^4$ with respect to $d = 3$ is $k_0 = 1$. This example illustrates one important result: a sequence is irregular if certain element is much larger than the others.

Example 2.4 Consider a sequence $\{c_i\}_{i=1}^4 = \{10, 10, 1, 1\}$ and $d = 2$ or 3. When $d = 2$, we have $10^2 < 1/2(10^2 + 10^2 + 1 + 1)$. Hence $\{c_i\}_{i=1}^4$ is regular with respect to $d = 2$. When $d = 3$, we have $10^2 > 1/3(10^2 + 10^2 + 1 + 1)$, $10^2 > 1/2(10^2 + 1 + 1)$ and $1 < 1/1(1 + 1)$. Hence $\{c_i\}_{i=1}^4$ is irregular with respect to $d = 3$ and the irregularity is $k_0 = 2$. This example shows that a sequence may be regular with respect to one integer but irregular with respect to another.

The minimizers of the frame potential in (3) are characterized by the following lemma (Casazza et al. 2006), which will be used to prove the necessary and sufficient conditions of optimal placements.

Lemma 2.5: *In $\mathbb{R}^d$, given a positive non-increasing sequence $\{c_i\}_{i=1}^n$ with irregularity as k_0, if the norms of the frame $\{\varphi_i\}_{i=1}^n$ are specified as $\|\varphi_i\| = c_i$ for all $i \in \{1, \ldots, n\}$, any minimizer of the frame potential in* (3) *is of the form*

$$\{\varphi_i\}_{i=1}^n = \{\varphi_i\}_{i=1}^{k_0} \cup \{\varphi_i\}_{i=k_0+1}^n,$$

where $\{\varphi_i\}_{i=1}^{k_0}$ is an orthogonal set, and $\{\varphi_i\}_{i=k_0+1}^n$ is a tight frame in the orthogonal complement of the span of $\{\varphi_i\}_{i=1}^{k_0}$. Any local minimizer is also a global minimizer.

From Lemma 2.5, a minimizer of the frame potential consists of an orthogonal set $\{\varphi_i\}_{i=1}^{k_0}$ and a tight frame $\{\varphi_i\}_{i=k_0+1}^n$. The partition of the two sets is determined by the irregularity of the specified frame norms $\{c_i\}_{i=1}^n$. When the irregularity $k_0 = 0$, it is clear that a minimizer of the frame potential is a tight frame. As a corollary of Lemma 2.5, the following result (Casazza et al. 2006) gives the existence condition of the solutions to (2).

Lemma 2.6: *In $\mathbb{R}^d$, given a positive sequence $\{c_i\}_{i=1}^n$, there exists a tight frame $\{\varphi_i\}_{i=1}^n$ with $\|\varphi_i\| = c_i$ for all $i \in \{1, \ldots, n\}$ solving* (2) *if and only if $\{c_i\}_{i=1}^n$ is regular.*

3 Problem Formulation

Consider one target and n sensors in $\mathbb{R}^d$ ($d = 2$ or 3 and $n \geq d$). The n sensors are of one of the following sensor types: bearing-only, range-only, and RSS. Sensor networks with mixed sensor types are not considered in this paper. Following Bishop et al. (2010), Doğançay and Hmam (2008), Martínez and Bullo (2006), Bishop and Jensfelt (2009), we assume that an initial target position estimate $p \in \mathbb{R}^d$ is available. The optimal placement will be determined based on this initial estimate. Denote the position of sensor i as $s_i \in \mathbb{R}^d$, $i \in \{1, \ldots, n\}$. Then $r_i = s_i - p$ denotes the position of sensor i relative to the target. The sensor-target placement can be fully described by $\{r_i\}_{i=1}^n$. Our aim is to determine the optimal $\{r_i\}_{i=1}^n$ such that certain objective function can be optimized. The distance between sensor i and the target is given by $\|r_i\|$. The unit-length vector

$$g_i = \frac{r_i}{\|r_i\|}$$

represents the bearing of sensor i relative to the target.

3.1 *Sensor Measurement Model and FIM*

For any sensor type in Table 1, the measurement model of sensor i can be expressed as

$$z_i = h_i(r_i) + v_i,$$

Table 1.: Measurement models and FIMs of the three sensor types.

Sensor type	Measurement model	FIM	Coefficient	Optimality criterion
Bearing-only	$h_i(r_i) = \dfrac{r_i}{\Vert r_i \Vert}$	$F = \sum_{i=1}^{n} c_i^2 (I_d - g_i g_i^T)$	$c_i = \dfrac{1}{\sigma_i \Vert r_i \Vert}$	$\min \left\Vert \sum_{i=1}^{n} c_i^2 g_i g_i^T \right\Vert^2$
Range-only	$h_i(r_i) = \Vert r_i \Vert$	$F = \sum_{i=1}^{n} c_i^2 g_i g_i^T$	$c_i = \dfrac{1}{\sigma_i}$	$\min \left\Vert \sum_{i=1}^{n} c_i^2 g_i g_i^T \right\Vert^2$
RSS	$h_i(r_i) = \ln \Vert r_i \Vert$	$F = \sum_{i=1}^{n} c_i^2 g_i g_i^T$	$c_i = \dfrac{1}{\sigma_i \Vert r_i \Vert}$	$\min \left\Vert \sum_{i=1}^{n} c_i^2 g_i g_i^T \right\Vert^2$

where $z_i \in \mathbb{R}^m$ denotes the measurement of sensor i, the function $h_i(r_i) : \mathbb{R}^d \to \mathbb{R}^m$ is determined by the sensor type as shown in Table 1, and $v_i \in \mathbb{R}^m$ is the additive measurement noise. We assume v_i to be a zero-mean Gaussian noise with covariance as $\Sigma_i = \sigma_i^2 I_m \in \mathbb{R}^{m \times m}$, where I_m denotes the $m \times m$ identity matrix. By further assuming the measurement noises of different sensors are uncorrelated, the FIM given by n sensors is expressed as

$$F = \sum_{i=1}^{n} \left(\frac{\partial h_i}{\partial p} \right)^T \Sigma_i^{-1} \frac{\partial h_i}{\partial p}, \tag{6}$$

where $\partial h_i / \partial p$ denotes the Jacobian of $h_i(r_i) = h_i(s_i - p)$ with respect to p. For a detailed derivation of the FIM formula in (6), we refer to (Bishop et al. 2010, Section 3).

The measurement models of bearing-only, range-only, and RSS sensors are given in Table 1. The measurement of a bearing-only sensor is conventionally modeled as one angle (azimuth) in 2D or two angles (azimuth and altitude) in 3D. The drawback of this kind of model is that the model complexity increases dramatically as the dimension increases. As a result, this conventional model is not suitable for analyzing 3D optimal placements. Note that a unit-length vector essentially characterizes a bearing and is very suitable to represent a bearing-only measurement. Thus we model the measurement of a bearing-only sensor as a *unit-length vector* pointing from the target to the sensor. As will be shown later, this new bearing-only measurement model will greatly simplify the formulation of optimal bearing-only placement problems in 2D and 3D. The measurement model of range-only sensors in Table 1 is the same as the one given by Bishop et al. (2010). The measurement model of RSS sensors in Table 1 is a modified version of the one in Bishop and Jensfelt (2009). Without loss of generality, we simplify the model in Bishop and Jensfelt (2009) by omitting certain additive and multiplicative constants.

By substituting $h_i(r_i)$ into (6), we can calculate the FIMs of the three sensor types. The calculation is straightforward and omitted here. The FIMs have been calculated and given in Table 1. As will be shown later, the coefficients $\{c_i\}_{i=1}^n$ in the FIM are crucial for determining optimal placements. Following Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009), Martínez and Bullo (2006), we assume the coefficient c_i to be *arbitrary but fixed.* (i) For bearing-only or RSS sensors, as $c_i = 1/(\sigma_i \Vert r_i \Vert)$, both σ_i and $\Vert r_i \Vert$ are assumed to be fixed. Otherwise, if $\Vert r_i \Vert$ is unconstrained, the placement will be optimal when $\Vert r_i \Vert$ approaches zero. To avoid this trivial solution, it is reasonable to assume $\Vert r_i \Vert$ to be fixed. (ii) For range-only sensors, as $c_i = 1/\sigma_i$, only σ_i is assumed to be fixed. Hence $\Vert r_i \Vert$ will have no influence on the optimality of the placements for range-only sensors.

To end this subsection, we would like to point out that the FIMs given in Table 1 are consistent with the ones given in Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009), Martínez and Bullo (2006) in 2D cases. To verify that, we can substitute $g_i = [\cos \theta_i, \sin \theta_i]^T \in \mathbb{R}^2$ into the FIMs in Table 1.

3.2 *A New Criterion for Optimal Placement*

The existing work on optimal senor placement has adopted various objective functions such as $\det F$, $\operatorname{tr} F$, and $\operatorname{tr} F^{-1}$. These objective functions are respectively referred as D-, T-, and A-optimality criteria in the field of optimal experimental design (Pukelsheim 1993). The most popular criterion used for optimal sensor placement is to maximize $\det F$, which can be interpreted as minimizing the volume of the uncertainty ellipsoid characterized by F^{-1}. However, this criterion is not suitable for analyzing optimal placements in 3D space because $\det F$ is hardly analytically tractable in $\mathbb{R}^3$. In order to analytically characterize optimal placements in $\mathbb{R}^2$ and $\mathbb{R}^3$, we next introduce a new criterion that is closely related to the conventional one.

Denote $\{\lambda_i\}_{i=1}^d$ as the eigenvalues of F. Let $\bar{\lambda} = 1/d\sum_{i=1}^d \lambda_i$. Since $\sum_{i=1}^d \lambda_i = \operatorname{tr} F$, it is easy to examine that $\bar{\lambda}$ is an invariant quantity for any F given in Table 1. In this paper, we will minimize the new objective function $\|F - \bar{\lambda} I_d\|^2$, which is of strong analytical tractability. Note $\|F - \bar{\lambda} I_d\|^2 = \sum_{i=1}^d (\lambda_i - \bar{\lambda})^2$. Hence minimizing $\|F - \bar{\lambda} I_d\|^2$ actually is to minimize the diversity of the eigenvalues of F. The following result shows that the new criterion has a close connection with the conventional one.

Lemma 3.1: *For any one of the three sensor types given in Table 1, we have*

$$\det F \le \bar{\lambda}^d,$$

where the equality holds if and only if

$$\|F - \bar{\lambda} I_d\|^2 = 0.$$

Proof For any one of the three sensor types, the FIM F is symmetric positive (semi) definite. Hence λ_j is real and nonnegative. From the FIMs shown in Table 1, we have $\sum_{j=1}^d \lambda_j = \operatorname{tr} F = (d-1)\sum_{i=1}^n c_i^2$ for bearing-only sensors, and $\sum_{j=1}^d \lambda_j = \operatorname{tr} F = \sum_{i=1}^n c_i^2$ for range-only or RSS sensors. Note $\{c_i\}_{i=1}^n$ is assumed to be fixed. Hence $\sum_{j=1}^d \lambda_j$ is an invariant quantity. By the inequality of arithmetic and geometric means, the conventional objective function $\det F$ satisfies

$$\det F = \prod_{j=1}^d \lambda_j \le \left(\frac{1}{d} \sum_{j=1}^d \lambda_j \right)^d = \bar{\lambda}^d,$$

where the equality holds if and only if $\lambda_j = \bar{\lambda}$ for all $j \in \{1, \dots, d\}$, which means

$$F = \bar{\lambda} I_d \Longleftrightarrow \|F - \bar{\lambda} I_d\|^2 = 0.$$

In short, $\det F$ is maximized to its upper bound $\bar{\lambda}^d$ if and only if $\|F - \bar{\lambda} I_d\|^2 = 0$. □

Loosely speaking, Lemma 3.1 suggests that minimizing $\|F - \bar{\lambda} I_d\|^2$ is equivalent to maximizing $\det F$. We next further examine the relationship between the new and conventional criterions case by case.

i) In $\mathbb{R}^2$, we have $\det F = 1/2((\operatorname{tr} \mathcal{F})^2 - \operatorname{tr}(F^2)) = 1/2\left(4\bar{\lambda}^2 - \|F\|^2\right)$ and $\|F - \bar{\lambda} I_2\|^2 = \operatorname{tr}(F - \bar{\lambda} I_2)^2 = \|F\|^2 - 2\bar{\lambda}^2$, which suggest

$$\|F - \bar{\lambda} I_2\|^2 = -2\det F + 2\bar{\lambda}^2.$$

Because $2\bar{\lambda}^2$ is constant, minimizing $\|F - \bar{\lambda} I_2\|^2$ is rigorously equivalent to maximizing $\det F$ in $\mathbb{R}^2$. As a result, our analysis based on the new criterion will be consistent with the 2D results in Bishop et al. (2010), Doğançay and Hmam (2008), Martínez and Bullo (2006), Bishop and Jensfelt (2009).

ii) In $\mathbb{R}^3$, if $\|F-\bar{\lambda}I_3\|^2$ is able to achieve zero, then $\det F$ can be maximized to its upper bound as shown in Lemma 3.1. In this case the new criterion is still rigorously equivalent to the conventional one.

iii) In $\mathbb{R}^3$, $\|F-\bar{\lambda}I_3\|^2$ is not able to reach zero in certain *irregular* cases (see Section 4 for the formal definition of irregular). In these cases $\det F$ and $\|F-\bar{\lambda}I_3\|^2$ may not be optimized simultaneously. But as will be shown later, the analysis of irregular cases in $\mathbb{R}^3$ based on the new criterion is a reasonable extension of the analysis of irregular cases in $\mathbb{R}^2$.

3.3 *Problem Statement*

We now formally state the optimal sensor placement problem that we are going to solve.

Problem 3.2: *Consider one target and n sensors in $\mathbb{R}^d$ ($d=2$ or 3 and $n \geq d$). The sensors involve only one of the three sensor types in Table 1. Given arbitrary but fixed positive coefficients $\{c_i\}_{i=1}^n$, find the optimal placement $\{g_i^*\}_{i=1}^n$ such that*

$$\{g_i^*\}_{i=1}^n = \underset{\{g_i\}_{i=1}^n \subset \mathbb{S}^{d-1}}{\arg\min} \|F-\bar{\lambda}I_d\|^2, \tag{7}$$

where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

Remark 2: The sensor-target placement can be fully described by $\{r_i\}_{i=1}^n$. Recall $\|r_i\|$ is assumed to be fixed for bearing-only or RSS sensors, and $\|r_i\|$ has no effect on the placement optimality for range-only sensors. Thus for any sensor type, the optimal sensor placement can also be fully described by $\{g_i\}_{i=1}^n$. That means we only need to determine the optimal sensor-target bearings $\{g_i^*\}_{i=1}^n$ to obtain the optimal placement.

Although the FIMs of different sensor types may have different formulas as shown in Table 1, the following result shows that substituting the FIMs of the three sensor types into (7) will lead to an identical objective function. The following result is important because it enables us to *unify* the formulations of optimal sensor placement for the three sensor types.

Lemma 3.3: *Consider one target and n sensors in $\mathbb{R}^d$ ($d=2$ or 3 and $n \geq d$). The sensors involve only one of the three sensor types in Table 1. The problem defined in (7) is equivalent to*

$$\{g_i^*\}_{i=1}^n = \underset{\{g_i\}_{i=1}^n \subset \mathbb{S}^{d-1}}{\arg\min} \|G\|^2, \tag{8}$$

where $G=\sum_{i=1}^n c_i^2 g_i g_i^T$.

Proof If all sensors are bearing-only, the FIM is $F=\sum_{i=1}^n c_i^2(I_d-g_ig_i^T)$ and then $\bar{\lambda} = 1/d\sum_{j=1}^d \lambda_j = \operatorname{tr} F/d = (d-1)/d\sum_{i=1}^n c_i^2$. Hence

$$\begin{aligned}\|F-\bar{\lambda}I_d\| &= \left\|\sum_{i=1}^n c_i^2(I_d-g_ig_i^T) - \frac{d-1}{d}\sum_{i=1}^n c_i^2 I_d\right\| \\ &= \left\|-\sum_{i=1}^n c_i^2 g_ig_i^T + \frac{1}{d}\sum_{i=1}^n c_i^2 I_d\right\|.\end{aligned}$$

If all sensors are range-only or RSS, the FIM is $F=\sum_{i=1}^n c_i^2 g_ig_i^T$ and then $\bar{\lambda}=1/d\sum_{j=1}^d \lambda_j =$

$\operatorname{tr} F/d = 1/d \sum_{i=1}^n c_i^2$. Hence

$$\|F - \bar{\lambda} I_d\| = \left\| \sum_{i=1}^n c_i^2 g_i g_i^T - \frac{1}{d} \sum_{i=1}^n c_i^2 I_d \right\|.$$

Therefore, for any sensor type in Table 1, the new objective function can be rewritten as

$$\begin{aligned} \|F - \bar{\lambda} I_d\|^2 &= \left\| \sum_{i=1}^n c_i^2 g_i g_i^T - \frac{1}{d} \sum_{i=1}^n c_i^2 I_d \right\|^2 \\ &= \|G\|^2 - \frac{1}{d} \left(\sum_{i=1}^n c_i^2 \right)^2. \end{aligned} \tag{9}$$

Because $1/d(\sum_{i=1}^n c_i^2)^2$ is constant, minimizing $\|G\|^2$ is equivalent to minimizing $\|F - \bar{\lambda} I_d\|^2$. □

One primary aim of this work is to solve the parameter optimization problem (8). It should be noted that we must clearly know the type of the sensors such that the coefficients $\{c_i\}_{i=1}^n$ in G can be calculated correctly. Once $\{c_i\}_{i=1}^n$ are calculated, the sensor types will be transparent to us. As a consequence, the analysis of optimal sensor placement in the sequel of the paper will apply to all the three sensor types.

Remark 3 : In this work, we only consider homogeneous sensor networks. But it is worthwhile noting that Lemma 3.3 actually is also valid for a heterogeneous sensor network which contains both range-only and RSS sensors. That is because the FIMs of the two sensor types have the same formula, and the total FIM would simply be the sum of the two respective FIMs of range-only and RSS sensors. As a result, the analysis in the rest of this paper also applies to heterogeneous sensor networks that contain both range-only and RSS sensors. In the heterogeneous case, the coefficient c_i should be calculated correctly according to the type of sensor i.

3.4 *Equivalent Sensor Placements*

Before solving (8), we identify a group of placements that result in the same value of $\|G\|^2$.

Proposition 3.4: *The objective function $\|G\|^2$ is invariant to the sign of g_i for all $i \in \{1, \dots, n\}$ and any orthogonal transformations over $\{g_i\}_{i=1}^n$.*

Proof First, $g_i g_i^T = (-g_i)(-g_i)^T$ for all $i \in \{1, \dots, n\}$, hence $\|G\|^2$ is invariant to the sign of g_i. Second, let $U \in \mathbb{R}^{d \times d}$ be an orthogonal matrix satisfying $U^T U = I_d$. Applying U to $\{g_i\}_{i=1}^n$ yields $\{g_i' = U g_i\}_{i=1}^n$. Then we have $G' = \sum_{i=1}^n c_i^2 g_i' (g_i')^T = \sum_{i=1}^n c_i^2 (U g_i)(U g_i)^T = U G U^T$. Since G and G' are both symmetric, we have $\|G'\|^2 = \operatorname{tr}(U G U^T U G U^T) = \operatorname{tr}(G^2) = \|G\|^2$. □

Geometrically speaking, changing the sign of g_i means flipping sensor i about the target, and an orthogonal transformation represents a rotation, reflection or both combined operation over all sensors. Therefore, Proposition 3.4 implies that these geometric operations cannot affect the value of $\|G\|^2$. Furthermore, it is straightforward to examine that $\det F$ is also invariant to these geometric operations. It is noticed that the invariance to the sign change of g_i was originally recognized in Doğançay and Hmam (2008) for 2D bearing-only sensor placements. By Proposition 3.4, we define the following equivalence relationship.

Definition 3.5 (Equivalent placements): Given arbitrary but fixed coefficients $\{c_i\}_{i=1}^n$, two placements $\{g_i\}_{i=1}^n$ and $\{g_i'\}_{i=1}^n$ are called equivalent if they are differed by indices permutation, flipping any sensors about the target, or any global rotation, reflection or both combined over all sensors.

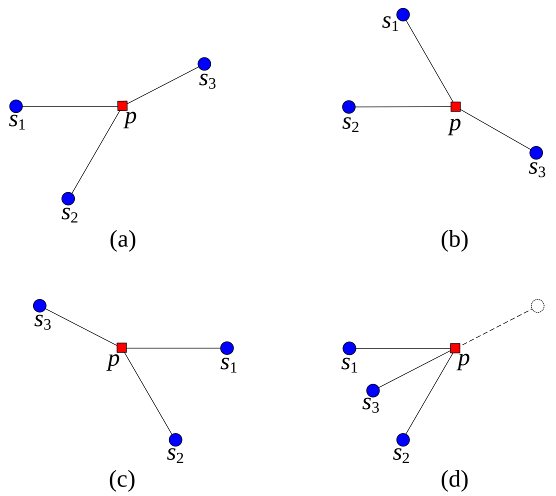


Figure 1.: Examples of equivalent placements ($d = 2, n = 3$): (a) Original placement. (b) Rotate all sensors about the target 60 degrees clockwise. (c) Reflect all sensors about the vertical axis. (d) Flipping the sensor s_3 about the target.

Due to the equivalence, there always exist an infinite number of equivalent optimal placements minimizing $\|G\|^2$. If two optimal placements are equivalent, they lead to the same objective function value. But the converse statement is not true in general. In Section 5.3, we will give the condition under which the converse is true. Examples of 2D equivalent placements are given in Figure 1.

4 Necessary and Sufficient Conditions for Optimal Placement

In this section, we prove the necessary and sufficient conditions for optimal placements solving (8). Recall $G = \sum_{i=1}^{n} c_i^2 g_i g_i^T$. Then we have

$$\begin{aligned}\|G\|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}(c_i c_j g_i^T g_j)^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(\varphi_i \varphi_j)^2,\end{aligned}$$

where $\varphi_i = c_i g_i$ and $\|\varphi_i\| = c_i$ for any $i \in \{1, \dots, n\}$. The vectors $\{\varphi_i\}_{i=1}^n$ actually form a frame in $\mathbb{R}^d$. Then the objective function $\|G\|^2$ is the frame potential of $\{\varphi_i\}_{i=1}^n$ as shown in (3), and the matrix G is the frame operator. Furthermore, since $\|\varphi_i\| = c_i$, the coefficient sequence $\{c_i\}_{i=1}^n$ will fully determine the minimizers of $\|G\|^2$. According to the irregularity of $\{c_i\}_{i=1}^n$, optimal placements can be categorized as regular and irregular as shown below.

When $\{c_i\}_{i=1}^n$ is regular, the necessary and sufficient condition of optimal placement is given below. The 2D version of the following result has been proposed in Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009).

Theorem 4.1 (Regular optimal placement): *In $\mathbb{R}^d$ with $d = 2$ or 3, if the positive coefficient sequence $\{c_i\}_{i=1}^n$ is regular, then the objective function $\|G\|^2$ satisfies*

$$\|G\|^2 \geq \frac{1}{d}\left(\sum_{i=1}^{n} c_i^2\right)^2. \tag{10}$$

The lower bound of $\|G\|^2$ *is achieved if and only if*

$$\sum_{i=1}^{n} c_i^2 g_i g_i^T = \frac{1}{d} \sum_{i=1}^{n} c_i^2 I_d. \tag{11}$$

Proof Let $\{\mu_j\}_{j=1}^d$ be the eigenvalues of G. Then $\sum_{j=1}^d \mu_j = \operatorname{tr} G = \sum_{i=1}^n c_i^2$ is constant. Let $\bar{\mu} = 1/d \sum_{j=1}^d \mu_j = 1/d \sum_{i=1}^n c_i^2$. It is obvious that

$$\|G\|^2 = \sum_{j=1}^{d} \mu_j^2 \geq d\bar{\mu}^2 = \frac{1}{d} \left(\sum_{i=1}^{n} c_i^2 \right)^2. \tag{12}$$

The lower bound of $\|G\|^2$ is achieved if and only if $\mu_j = \bar{\mu}$ for all $j \in \{1, \ldots, d\}$, which implies $G = \bar{\mu} I_d$ and hence equation (11). By denoting $\varphi_i = c_i g_i$, equation (11) becomes $\sum_{i=1}^n \varphi_i \varphi_i^T = 1/d \sum_{i=1}^n c_i^2 I_d$ which is the same as (2). Thus a regular optimal placement solving (11) corresponds to a tight frame. Because $\{c_i\}_{i=1}^n$ is regular, by Lemma 2.6 there exist optimal placements solving (11). □

We call a placement *regular* when its coefficient sequence is regular, and *regular optimal* when it solves (11). To obtain a regular optimal placement, we need further to solve (11). Details of the solutions to (11) will be given in Section 5.1. When $\|G\|^2$ reaches its lower bound given in (12), it is straightforward to see $\|F - \bar{\lambda} I_d\|^2 = 0$ by (9). Thus by Lemma 3.1, the conventional objective function $\det F$ would also be maximized to its upper bound. Then we have the following result.

Corollary 4.2: *In* $\mathbb{R}^d$ *with* $d = 2$ *or* 3*, a regular optimal placement not only minimizes the new objective functions* $\|G\|^2$ *and* $\|F - \bar{\lambda} I_d\|^2$*, but also maximizes the conventional one* $\det F$*.*

When $\{c_i\}_{i=1}^n$ is irregular, (11) will have no solution by Lemma 2.6. Then the the necessary and sufficient condition of optimal placement is given below. The 2D version of the following result has been proposed in Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009).

Theorem 4.3 (Irregular optimal placement): *In* $\mathbb{R}^d$ *with* $d = 2$ *or* 3*, if the positive coefficient sequence* $\{c_i\}_{i=1}^n$ *is irregular with irregularity as* $k_0 \geq 1$*, without loss of generality* $\{c_i\}_{i=1}^n$ *can be assumed to be a non-increasing sequence, and then the objective function* $\|G\|^2$ *satisfies*

$$\|G\|^2 \geq \sum_{i=1}^{k_0} c_i^4 + \frac{1}{d - k_0} \left(\sum_{i=k_0+1}^{n} c_i^2 \right)^2. \tag{13}$$

The lower bound of $\|G\|^2$ *is achieved if and only if*

$$\{g_i\}_{i=1}^n = \{g_i\}_{i=1}^{k_0} \cup \{g_i\}_{i=k_0+1}^n, \tag{14}$$

where $\{g_i\}_{i=1}^{k_0}$ *is an orthogonal set, and* $\{g_i\}_{i=k_0+1}^n$ *forms a regular optimal placement in the* $(d - k_0)$*-dimensional orthogonal complement of* $\{g_i\}_{i=1}^{k_0}$*.*

Proof Recall $\|G\|^2$ is the frame potential of the frame $\{\varphi_i\}_{i=1}^n$ where $\varphi_i = c_i g_i$. From Lemma 2.5, the minimizer of $\|G\|^2$ is of the following form: $\{c_i g_i\}_{i=1}^{k_0}$ is an orthogonal set, and $\{c_i g_i\}_{i=k_0+1}^n$ is a tight frame (i.e., a regular optimal placement) in the orthogonal complement of $\{c_i g_i\}_{i=1}^{k_0}$.

Let $\Phi_1 = [\varphi_1, \ldots, \varphi_{k_0}] \in \mathbb{R}^{d \times k_0}$, $\Phi_2 = [\varphi_{k_0+1}, \ldots, \varphi_n] \in \mathbb{R}^{d \times (n-k_0)}$, and $\Phi = [\Phi_1, \Phi_2] \in \mathbb{R}^{d \times n}$.

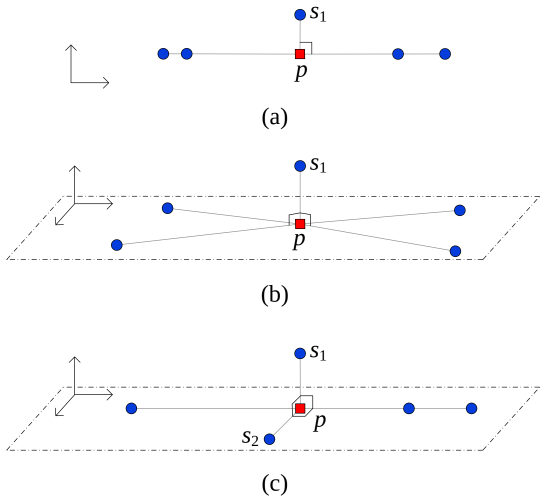


Figure 2.: An illustration of the three kinds of irregular optimal placements in $\mathbb{R}^2$ and $\mathbb{R}^3$. (a) $d = 2$, $k_0 = 1$; (b) $d = 3$, $k_0 = 1$; (c) $d = 3$, $k_0 = 2$.

When $\{g_i\}_{i=1}^n$ is of the form in (14), the columns of Φ_1 are orthogonal to those of Φ_2. Then

$$\|G\|^2 = \operatorname{tr}(\Phi^T\Phi)^2 = \operatorname{tr}(\Phi_1^T\Phi_1)^2 + \operatorname{tr}(\Phi_2^T\Phi_2)^2.$$

Because $\{g_i\}_{i=1}^{k_0}$ is an orthogonal set, we have $\operatorname{tr}(\Phi_1^T\Phi_1)^2 = \sum_{i=1}^{k_0}\|\varphi_i\|^4 = \sum_{i=1}^{k_0} c_i^4$. Because $\{g_i\}_{i=k_0+1}^n$ is a regular optimal placement in a $(d-k_0)$-dimensional subspace, we have $\operatorname{tr}(\Phi_2^T\Phi_2)^2 = 1/(d-k_0)(\sum_{i=k_0+1}^n c_i^2)^2$ by Theorem 4.1. Therefore, when $\{g_i\}_{i=1}^n$ is of the form in (14), the objective function $\|G\|^2$ reaches its lower bound as shown in (13). □

We call a placement *irregular* when its coefficient sequence is irregular, and *irregular optimal* when it is of the form in (14). In Theorem 4.3, $\{g_i\}_{i=k_0+1}^n$ is a regular optimal placement in a $(d-k_0)$-dimensional space. Thus Theorem 4.3 implies that an irregular optimal placement problem would be eventually converted to a regular one in a lower dimensional subspace.

As shown by Theorems 4.3, the irregularity of $\{c_i\}_{i=1}^n$ plays a key role in determining optimal placements. Recall the irregularity k_0 of an irregular sequence with respect to d satisfies $1 \le k_0 \le d-1$. As $d = 2$ or 3 in our work, it is possible to enumerate all the kinds of irregular optimal placements. Specifically, in $\mathbb{R}^2$, we have $d = 2$ and hence $k_0 = 1$; in $\mathbb{R}^3$, we have $d = 3$ and hence $k_0 = 1$ or 2. Thus there exist only *three* kinds of irregular optimal placements in $\mathbb{R}^2$ and $\mathbb{R}^3$. By Theorem 4.3, these three kinds of irregular optimal placements can be intuitively described as below.

i) Irregular optimal placement in $\mathbb{R}^2$ with irregularity $k_0 = 1$: the vector g_1 is orthogonal to $\{g_i\}_{i=2}^n$, and $\{g_i\}_{i=2}^n$ are collinear. See an illustration in Figure 2 (a).
ii) Irregular optimal placement in $\mathbb{R}^3$ with irregularity $k_0 = 1$: the vector g_1 is orthogonal to $\{g_i\}_{i=2}^n$, and $\{g_i\}_{i=2}^n$ form a regular optimal placement in the 2D plane perpendicular to g_1. See an illustration in Figure 2 (b).
iii) Irregular optimal placement in $\mathbb{R}^3$ with irregularity $k_0 = 2$: the vectors g_1, g_2 and $\{g_i\}_{i=3}^n$ are mutually orthogonal, and $\{g_i\}_{i=3}^n$ are collinear. See an illustration in Figure 2 (c).

Up to this point, Theorems 4.1 and 4.3 clearly indicate that the cruciality of the coefficients $\{c_i\}_{i=1}^n$ in determining optimal sensor placements. The coefficient c_i actually is the weight for sensor i. The larger the weight c_i is, the more sensor i contributes to the FIM. Recall $c_i = 1/\sigma_i$ for range-only sensors, and $c_i = 1/(\sigma_i\|r_i\|)$ for bearing-only or RSS sensors. Hence for range-only sensors, the measurement noise level of a sensor can affect its weight; for bearing-only or RSS sensors, both measurement noise level and sensor-target range can affect the weight of a sensor and hence the optimal placement. In addition, a sequence $\{c_i\}_{i=1}^n$ is irregular only if certain c_i's are much larger than the others. Since large c_i implies small σ_i (and small $\|r_i\|$), a placement is irregular only if certain sensors can give much more accurate measurements (and are much closer to the target) than the others.

To make our analysis more general, we do not assume σ_i's to be identical in this work. But it is also meaningful to check the special case that $\sigma_i = \sigma_j$ for all $i \neq j$. First, for bearing-only or RSS-based sensors, the coefficient is $c_i = 1/(\sigma_i \|r_i\|)$. Then when $\sigma_i = \sigma_j$ for all $i \neq j$, from the fundamental inequality (5), a regular sequence $\{c_i\}_{i=1}^n$ implies

$$\max_{j=1,\ldots,n} \frac{1}{\|r_j\|^2} \leq \frac{1}{d} \sum_{i=1}^{n} \frac{1}{\|r_i\|^2}, \tag{15}$$

which geometrically means no sensor is much closer to the target than the others. The 2D version of inequality (15) has been proposed in Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009). Second, for range-only sensors, the coefficient is $c_i = 1/\sigma_i$. If $\sigma_i = \sigma_j$ for all $i \neq j$, then $c_i = c_j$. Hence $\{c_i\}_{i=1}^n$ is regular with respect to any $d \leq n$ as shown in Example 2.2.

We next consider an important special case $n = d$, i.e., the sensor number equals to the dimension of the space. This case is important because the optimal placement will be independent to the coefficients $\{c_i\}_{i=1}^n$ in this case. The optimal placement in the case of $n = d = 2$ has been solved by Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009), Martínez and Bullo (2006).

Theorem 4.4: *In $\mathbb{R}^d$ with $d = 2$ or 3, if $n = d$, the objective function $\|G\|^2$ satisfies*

$$\|G\|^2 \geq \sum_{i=1}^{d} c_i^4.$$

The lower bound of $\|G\|^2$ is achieved if and only if $\{g_i\}_{i=1}^d$ is an orthogonal basis of $\mathbb{R}^d$.

Proof Since $G = \sum_{i=1}^d c_i^2 g_i g_i^T$ and $g_i^T g_i = 1$ for all $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned}
\|G\|^2 &= \mathrm{tr}\,(G^2) \\
&= \sum_{i=1}^{d} \sum_{j=1}^{d} c_i^2 c_j^2 (g_i^T g_j)^2 \\
&= \sum_{i=1}^{d} \sum_{j=1, j\neq i}^{d} c_i^2 c_j^2 (g_i^T g_j)^2 + \sum_{i=1}^{d} c_i^4 \\
&\geq \sum_{i=1}^{d} c_i^4,
\end{aligned}$$

where the equality holds if and only if $g_i^T g_j = 0$ for all $i, j \in \{1, \ldots, d\}$ and $i \neq j$. □

Theorem 4.4 actually can be proved as a corollary of Theorems 4.1 and 4.3. But as shown above, we can also directly prove it in a straightforward way *without* employing frame theory. This can be explained from the point of view of redundancy. Recall the constant n/d reflects the redundancy of the system. When $n/d = 1$, the system has no redundancy and hence frames are no longer necessary for the optimality analysis.

5 Analytical Properties of Optimal Placements

In this section, we further explore a number of analytical properties of optimal placements in 2D and 3D. Theorem 4.3 implies that an irregular optimal placement problem can be eventually converted to a regular one in a lower dimensional space. Hence we will only focus on regular optimal placements.

5.1 *Explicit Construction*

Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009), Martínez and Bullo (2006) have proposed a number of methods to explicitly construct some special 2D optimal placements. However, the construction of generic optimal placements in 2D or 3D is still an open problem. In our work, as stated in Theorem 4.1, constructing a regular optimal placement that solves (11) is equivalent to constructing a tight frame. Thus we successfully convert the optimal sensor placement problem to a tight frame problem. Note tight frame construction has already been well studied in the literature on frame theory. Therefore, one can construct generic optimal placements (i.e., tight frames) of an arbitrary number of sensors in 2D or 3D by referring to the literature on tight frame construction. We will not discuss the construction of tight frames in detail here. Interested readers may refer to Feng et al. (2006), Casazza and Leon (2006), Cahill et al., Casazza et al. (2012), to name a few.

The necessary and sufficient condition for 2D optimal placements has already been proposed in Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009), where the sufficiency proof, however, is not given. Next we present a complete proof *without* employing frame theory. In the meantime, more importantly we propose an algorithm for explicitly constructing arbitrary 2D regular optimal placements. The following lemma can be found in Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009), Martínez and Bullo (2006), Benedetto and Fickus (2003), Goyal et al. (2001), Fickus (2001).

Lemma 5.1: *In $\mathbb{R}^2$, the unit-length vector g_i can be written as $g_i = [\cos\theta_i, \sin\theta_i]^T$. Then* (11) *is equivalent to*

$$\sum_{i=1}^{n} c_i^2 \bar{g}_i = 0, \tag{16}$$

where $\bar{g}_i = [\cos 2\theta_i, \sin 2\theta_i]^T$.

Proof Substituting $g_i = [\cos\theta_i, \sin\theta_i]^T$ into (11) gives

$$\sum_{i=1}^{n} c_i^2 \begin{bmatrix} \frac{1}{2}\cos 2\theta_i & \frac{1}{2}\sin 2\theta_i \\ \frac{1}{2}\sin 2\theta_i & -\frac{1}{2}\cos 2\theta_i \end{bmatrix} = 0,$$

which is equivalent to (16). □

By Lemma 5.1, the matrix equation (11) is simplified to a vector equation (16). In order to construct $\{g_i\}_{i=1}^n$ solving (11), we can first construct $\{\bar{g}_i\}_{i=1}^n$ solving (16). Once $\bar{g}_i = [\cos 2\theta_i, \sin 2\theta_i]^T$ is obtained, g_i can be retrieved as $g_i = \pm[\cos\theta_i, \sin\theta_i]^T$. Note the sign changes of g_i will give equivalent optimal placements as stated in Proposition 3.4.

Theorem 5.2: *In $\mathbb{R}^2$, given a positive sequence $\{c_i\}_{i=1}^n$, there exists $\{\bar{g}_i\}_{i=1}^n$ with $\|\bar{g}_i\| = 1$*

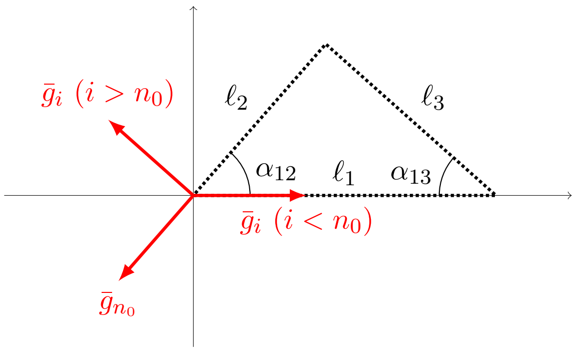


Figure 3.: A geometric illustration of Algorithm **1**.

solving (16) *if and only if*

$$\max_{j=1,\ldots,n} c_j^2 \le \frac{1}{2}\sum_{i=1}^{n} c_i^2. \tag{17}$$

Proof Necessity: If $\sum_{i=1}^n c_i^2\bar{g}_i = 0$, then $c_j^2\bar{g}_j = \sum_{i\neq j} c_i^2\bar{g}_i$ for all $j \in \{1,\ldots,n\}$. Hence $c_j^2 = \|c_j^2\bar{g}_j\| = \|\sum_{i\neq j} c_i^2\bar{g}_i\| \le \sum_{i\neq j}\|c_i^2\bar{g}_i\| = \sum_{i\neq j} c_i^2$. Then adding c_j^2 on both sides of the inequality gives $2c_j^2 \le \sum_{i=1}^n c_i^2$.

Sufficiency: If $c_j^2 \le 1/2\sum_{i=1}^n c_i^2$ for all $j \in \{1,\ldots,n\}$, it is obvious that there always exists an index n_0 $(2 \le n_0 \le n)$ such that

$$c_1^2 + \cdots + c_{n_0-1}^2 \le \frac{1}{2}\sum_{i=1}^{n} c_i^2, \tag{18}$$

$$c_1^2 + \cdots + c_{n_0-1}^2 + c_{n_0}^2 \ge \frac{1}{2}\sum_{i=1}^{n} c_i^2. \tag{19}$$

When $n_0 < n$, denote

$$\begin{aligned} \ell_1 &= c_1^2 + \cdots + c_{n_0-1}^2, \\ \ell_2 &= c_{n_0}^2, \\ \ell_3 &= c_{n_0+1}^2 + \cdots + c_n^2. \end{aligned} \tag{20}$$

Obviously $\ell_1 + \ell_2 + \ell_3 = \sum_{i=1}^n c_i^2$. From (17), $c_{n_0} \le 1/2\sum_{i=1}^n c_i^2$ and hence $\ell_1 + \ell_3 \ge \ell_2$. From (18), $\ell_1 \le 1/2\sum_{i=1}^n c_i^2$ and hence $\ell_2 + \ell_3 \ge \ell_1$. From (19), $\ell_1 + \ell_2 \ge 1/2\sum_{i=1}^n c_i^2$ and hence $\ell_1+\ell_2 \ge \ell_3$. Therefore, ℓ_1, ℓ_2 and ℓ_3 satisfy the triangle inequality and can form a triangle. Choose $\bar{g}_1 = \cdots = \bar{g}_{n_0-1}$. Then $\sum_{i=1}^{n_0-1} c_i^2\bar{g}_i = \ell_1\bar{g}_1$. Choose $\bar{g}_{n_0+1} = \cdots = \bar{g}_n$. Then $\sum_{i=n_0+1}^{n} c_i^2\bar{g}_i = \ell_3\bar{g}_n$. Then (16) becomes

$$\ell_1\bar{g}_1 + \ell_2\bar{g}_{n_0} + \ell_3\bar{g}_n = 0. \tag{21}$$

We can choose $\bar{g}_1, \bar{g}_{n_0}$ and $\bar{g}_n$ that align with the three sides of the triangle with side length as ℓ_1, ℓ_2 and ℓ_3, respectively (see Figure 3). Then (21) and consequently (16) can be solved. When $n_0 = n$, the above proof is still valid. In this case, we have $\ell_3 = 0$ and $\ell_1 = \ell_2$, and (21) becomes $\bar{g}_1 + \bar{g}_{n_0} = 0$. □

From the proof of Theorem 5.2, a method for explicitly constructing 2D regular optimal

Algorithm 1 Construction of 2D regular optimal placements $\{g_i\}_{i=1}^n$ with coefficients $\{c_i\}_{i=1}^n$.

1: Choose n_0 satisfying (18) and (19). Then compute ℓ_1, ℓ_2 and ℓ_3 in (20).
2: Compute interior angles α_{12} and α_{13} of the triangle with side lengths as ℓ_1, ℓ_2 and ℓ_3 (See Figure 3).
3: Choose $g_i = [1, 0]^T$ for $i \in \{1, \ldots, n_0 - 1\}$, $g_{n_0} = [\cos((\pi + \alpha_{12})/2), \sin((\pi + \alpha_{12})/2)]^T$, and $g_i = [\cos((\pi - \alpha_{13})/2), \sin((\pi - \alpha_{13})/2)]^T$ for $i \in \{n_0 + 1, \ldots, n\}$.

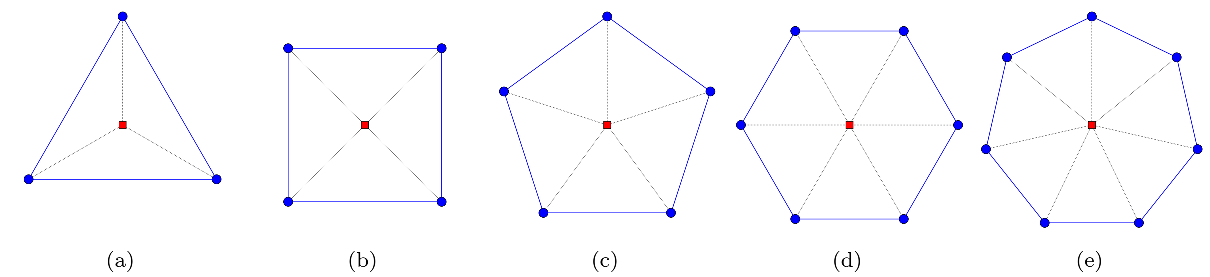


Figure 4.: Examples of 2D equally-weighted optimal placements: regular polygons. Red square: target; blue dots: sensors.

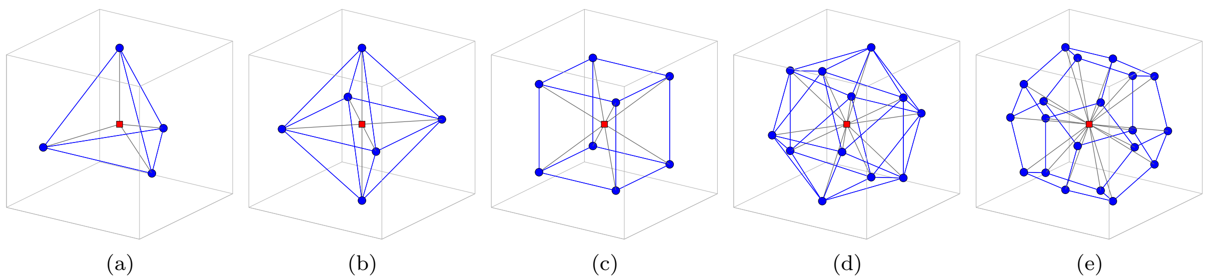


Figure 5.: Examples of 3D equally-weighted optimal placements: Platonic solids. Red square: target; blue dots: sensors. (a) Tetrahedron, $n = 4$. (b) Octahedron, $n = 6$. (c) Hexahedron, $n = 8$. (d) Icosahedron, $n = 12$. (e) Dodecahedron, $n = 20$.

placements can be summarized in Algorithm **1**. The following example illustrates Algorithm **1**.

Example 5.3 In $\mathbb{R}^2$, consider six bearing-only sensors with sensor-target ranges respectively as $\|r_1\| = 5$, $\|r_2\| = 6$, $\|r_3\| = 7$, $\|r_4\| = 8$, $\|r_5\| = 9$, and $\|r_6\| = 10$. The measurement noise variance is $\sigma_i = 1$ for all $i \in \{1, \ldots, 6\}$. Recall $c_i = 1/(\sigma_i \|r_i\|)$ for bearing-only sensors. Then $c_1^2 = 0.0400$, $c_2^2 = 0.0278$, $c_3^2 = 0.0204$, $c_4^2 = 0.0156$, $c_5^2 = 0.0123$, $c_6^2 = 0.0100$, and $1/2 \sum_{i=1}^6 c_i^2 = 0.0631$. It is easy to check the sequence $\{c_i\}_{i=1}^6$ is regular. Because $c_1^2 < 1/2 \sum_{i=1}^6 c_i^2$ and $c_1^2 + c_2^2 > 1/2 \sum_{i=1}^6 c_i^2$, choose $n_0 = 2$. Hence $\ell_1 = 0.0400$, $\ell_2 = 0.0278$, and $\ell_3 = 0.0584$. Then $\alpha_{12} = 2.0560\,\text{rad}$ and $\alpha_{13} = 0.4344\,\text{rad}$. As instructed in Algorithm **1**, choose $g_1 = [1, 0]^T$, $g_2 = [0.8563, -0.5165]^T$, $g_3 = \cdots = g_6 = [0.2155, 0.9765]^T$. Then it can be verified that $\sum_{i=1}^6 c_i^2 g_i g_i^T = 1/2 \sum_{i=1}^6 c_i^2 I_2$.

5.2 *Equally-weighted Optimal Placements*

The coefficient c_i actually is the weight for sensor i. Hence we call a placement *equally-weighted* if $c_1 = \cdots = c_n$. In the equally-weighted case, all sensors play equal roles for target localization. For bearing-only or RSS sensors, the placement is equally-weighted when $\sigma_i = \sigma_j$ and $\|r_i\| = \|r_j\|$ for all $i \neq j$ as $c_i = 1/(\sigma_i \|r_i\|)$. The corresponding geometry is that all sensors are restricted on a 2D circle or a 3D sphere centered at the target. For range-only sensors, the placement is equally-weighted when $\sigma_i = \sigma_j$ for all $i \neq j$ as $c_i = 1/\sigma_i$. As shown in Example 2.2, $\{c_i\}_{i=1}^n$ is regular with respect to any $d \leq n$ if $c_1 = \cdots = c_n$. Hence equally-weighted placements must be regular.

Equally-weighted placements are important because they often arise in practice and have some important special properties. In the equally-weighted case, (11) is simplified to $\sum_{i=1}^{n} g_i g_i^T = n/dI_d$, which implies that an equally-weighted optimal placement is essentially a unit-norm tight frame (Benedetto and Fickus 2003, Fickus 2001). In $\mathbb{R}^2$, an equally-weighted placement is optimal if n ($n \geq 3$) sensors are located at the vertices of an n-side regular polygon (Benedetto and Fickus 2003, Fickus 2001, Bishop et al. 2010, Doğançay and Hmam 2008, Bishop and Jensfelt 2009, Martínez and Bullo 2006) as shown in Figure 4. In $\mathbb{R}^3$, an equally-weighted placement is optimal if n sensors are located at the vertices of a Platonic solid (Benedetto and Fickus 2003, Fickus 2001). There are only five Platonic solids as shown in Figure 5. It should be noted that equally-weighted optimal placements are not limited to regular polygons or Platonic solids. In Section 5.4 we will show more examples of equally-weighted optimal placements.

5.3 *Uniqueness*

Due to placement equivalence, there exist at least an infinite number of equivalent optimal placements minimizing $\|G\|^2$. It is interesting to ask whether all optimal placements that minimize $\|G\|^2$ are mutually equivalent, or in other words, whether the optimal placement is *unique* up to the equivalence. We next give the conditions under which the answer is positive.

According to Theorem 4.4, it is clear that the optimal placement is unique in the case of $n = d$. We next show the regular optimal placement is also unique in the case of $n = d + 1$ (i.e., three sensors in $\mathbb{R}^2$ or four sensors in $\mathbb{R}^3$). The uniqueness will be proved by construction, which is inspired by the work in Goyal et al. (2001) on unit-norm tight frames.

Theorem 5.4: *In $\mathbb{R}^d$ with $d = 2$ or 3, if $n = d + 1$, given a regular coefficient sequence $\{c_i\}_{i=1}^{d+1}$, the regular optimal placement $\{g_i\}_{i=1}^{d+1}$ is unique up to the equivalence in Definition 3.5.*

Proof Suppose $\{g_i\}_{i=1}^{d+1}$ is a regular optimal placement solving (11). Denote $\varphi_i = c_i g_i$ and $\Phi = [\varphi_1, \ldots, \varphi_{d+1}] \in \mathbb{R}^{d\times(d+1)}$. Then (11) can be written in matrix form as $\Phi\Phi^T = 1/d \sum_{i=1}^{d+1} c_i^2 I_d$. Hence Φ has mutually orthogonal *rows* with row norm as $\sqrt{1/d \sum_{i=1}^{d+1} c_i^2}$. Let $x = [x_1, \ldots, x_{d+1}] \in \mathbb{R}^{d+1}$ be a vector in the orthogonal complement of the row space of Φ. Assume $\|x\| = \sqrt{1/d \sum_{i=1}^{d+1} c_i^2}$. Adding x^T after the last row of Φ yields an augmented matrix $\Phi_{\text{aug}} = \left[\Phi^T, x\right]^T \in \mathbb{R}^{(d+1)\times(d+1)}$. It is clear that $\Phi_{\text{aug}}\Phi_{\text{aug}}^T = \frac{1}{d}\sum_{i=1}^{d+1} c_i^2 I_{d+1}$. Thus Φ_{aug} is a scaled orthogonal matrix and its *columns* are mutually orthogonal. The jth column of Φ_{aug} is $[\varphi_j^T, x_j]^T \in \mathbb{R}^{d+1}$ for all $j \in \{1, \ldots, d+1\}$. Note the column norm of Φ_{aug} is $\sqrt{1/d \sum_{i=1}^{d+1} c_i^2}$. Then we have $\|\varphi_j\|^2 + x_j^2 = 1/d \sum_{i=1}^{d+1} c_i^2$ and hence

$$x_j = \pm\sqrt{\frac{1}{d}\sum_{i=1}^{d+1} c_i^2 - c_j^2}. \tag{22}$$

The regularity of $\{c_i\}_{i=1}^{d+1}$ ensures $1/d \sum_{i=1}^{d+1} c_i^2 - c_j^2 \geq 0$.

By reversing the above proof, we can obtain an explicit construction algorithm for optimal placement with $n = d+1$ as shown in Algorithm **2**. The rest is to prove the constructed optimal placements are mutually equivalent. First, given a vector $x \in \mathbb{R}^{d+1}$ satisfying (22), let Φ and Φ' be two different bases of the orthogonal complement of x. Due to orthogonality, there exists an orthogonal matrix $U \in \mathbb{R}^{(d+1)\times(d+1)}$ such that

$$U \begin{bmatrix} \Phi \\ x^T \end{bmatrix} = \begin{bmatrix} \Phi' \\ x^T \end{bmatrix}. \tag{23}$$

Algorithm 2 Construction of the unique regular optimal placement $\{g_i\}_{i=1}^{d+1}$ with coefficients $\{c_i\}_{i=1}^{d+1}$.

1: Choose $x = [x_1, \dots, x_{d+1}] \in \mathbb{R}^{d+1}$ with $x_j = \pm\sqrt{1/d \sum_{i=1}^{d+1} c_i^2 - c_j^2}$ for $i \in \{1, \dots, d+1\}$.

2: Use the singular value decomposition (SVD) to numerically compute an orthogonal basis of the orthogonal complement of x. Let $x = U\Sigma V^T$ be an SVD of x, where $U \in \mathbb{R}^{(d+1)\times(d+1)}$ is an orthogonal matrix.

3: Let u_i denote the ith column of U. Then $x = \pm\sqrt{1/d \sum_{i=1}^{d+1} c_i^2} u_1$, and Φ can be constructed as

$$\Phi = \sqrt{\frac{1}{d}\sum_{i=1}^{d+1} c_i^2}\, [u_2, \dots, u_{d+1}]^T \in \mathbb{R}^{d\times(d+1)}. \tag{25}$$

4: Compute $g_i = \varphi_i / c_i$ for $i \in \{1, \dots, d+1\}$.

Write U as

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \tag{24}$$

where $U_{11} \in \mathbb{R}^{d\times d}, U_{12} \in \mathbb{R}^{d\times 1}, U_{21} \in \mathbb{R}^{1\times d}$, and $U_{22} \in \mathbb{R}$. Substituting (24) into (23) gives $U_{21}\Phi + (U_{22} - 1)x^T = 0$. Since the rows of Φ and x^T are linearly independent, we have $U_{21} = 0$, $U_{22} = 1$. Thus $U_{12} = 0$ and $U_{11}\Phi = \Phi'$. Therefore, the placements described by Φ and Φ' are differed only by an orthogonal transformation U_{11}. From Definition 3.5, the two placements are equivalent. Second, let $E \in \mathbb{R}^{(d+1)\times(d+1)}$ be a diagonal matrix with diagonal entries as 1 or -1. Given arbitrary x and x' both satisfying (22), there exists an E such that $x' = Ex$. Note E is also an orthogonal matrix. It can be analogously proved that the optimal placements would be differed by an orthogonal transformation and a number of flipping of sensors about the target. From Definition 3.5, these placements are also equivalent. □

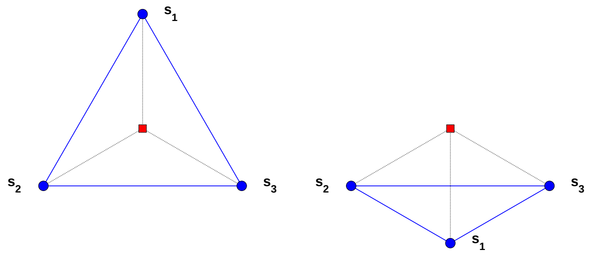
From the proof of Theorem 5.4, a method for explicitly constructing the unique regular optimal placement in the case of $n = d+1$ can be summarized as Algorithm **2**. The following example illustrates Algorithm **2**.

Example 5.5 In $\mathbb{R}^3$, consider four bearing-only sensors with sensor-target ranges respectively as $\|r_1\| = 20$, $\|r_2\| = 21$, $\|r_3\| = 22$, and $\|r_4\| = 23$. The measurement noise variance of the ith sensor is $\sigma_i = 0.01$ with $i \in \{1, \dots, 4\}$. Recall $c_i = 1/(\sigma_i \|r_i\|)$ for bearing-only sensors. Then $c_1^2 = 25.00$, $c_2^2 = 22.68$, $c_3^2 = 20.66$, $c_4^2 = 18.90$ and $1/3\sum_{i=1}^4 c_i^2 = 29.08$. The sequence $\{c_i\}_{i=1}^4$ is regular. From (22), choose $x = [2.02, 2.53, 2.90, 3.19]^T$. Compute the SVD of x and use (25) to compute Φ as

$$\Phi = \begin{bmatrix} -2.5307 & 4.5286 & -0.9906 & -1.0891 \\ -2.9016 & -0.9906 & 4.2568 & -1.2487 \\ -3.1901 & -1.0891 & -1.2487 & 4.0197 \end{bmatrix}.$$

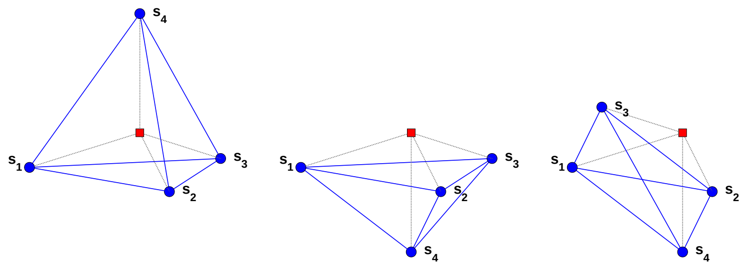
It can be verified $\sum_{i=1}^4 c_i^2 g_i g_i^T = \Phi\Phi^T = 1/3\sum_{i=1}^4 c_i^2 I_3$.

Figure 6 and Figure 7 show examples of unique optimal placements. Suppose all sensors have the same measurement noise standard deviation. Then the regular triangle in Figure 6 (a) is equally-weighted optimal as shown in Section 5.2. By Theorem 5.4, the regular triangle placement is also unique. Thus the two equivalent placements in Figure 6 represent all possible forms of the equally-weighted optimal placements with $n = 3$ in $\mathbb{R}^2$. Analogously, the three equivalent



(a) (b)

Figure 6.: The unique equally-weighted optimal placements with $n = 3$ in $\mathbb{R}^2$. Red square: target; blue dots: sensors. (a) Regular triangle. (b) Flip s_1 about the target.



(a) (b) (c)

Figure 7.: The unique equally-weighted optimal placements with $n = 4$ in $\mathbb{R}^3$. Red square: target; blue dots: sensors. (a) Regular tetrahedron. (b) Flip s_4 about the target. (c) Flip s_4 and s_3 about the target.

placements in Figure 7 present all possible forms of the equally-weighted optimal placements with $n = 4$ in $\mathbb{R}^3$.

When $n > d + 1$, the regular optimal placement may not be unique. In the next subsection, we will give examples to show the optimal placement may not be unique when $n \geq 4$ in $\mathbb{R}^2$ or $n \geq 6$ in $\mathbb{R}^3$. Now a question remains: whether the regular optimal placement with $n = 5$ in $\mathbb{R}^3$ is unique. The answer is negative. The following gives an explanation as well as an algorithm for explicitly constructing regular optimal placements with $n = 5$ in $\mathbb{R}^3$.

Suppose the sequence $\{c_i\}_{i=1}^5$ is regular with respect to $d = 3$. Denote $\varphi_i = c_i g_i$ and $\Phi = [\varphi_1, \ldots, \varphi_5] \in \mathbb{R}^{3\times 5}$. Then (11) becomes $\Phi\Phi^T = 1/3\sum_{i=1}^5 c_i^2 I_3$. There always exists $\Phi' = [\varphi'_1, \ldots, \varphi'_5] \in \mathbb{R}^{2\times 5}$ in the orthogonal complement of the row space of Φ such that

$$\begin{bmatrix} \Phi \\ \Phi' \end{bmatrix} \begin{bmatrix} \Phi^T \ \Phi'^T \end{bmatrix} = \frac{1}{3}\sum_{i=1}^{5} c_i^2 I_5,$$

which implies $\|\varphi_j\|^2 + \|\varphi'_j\|^2 = 1/3\sum_{i=1}^5 c_i^2$ and $\Phi'\Phi'^T = 1/3\sum_{i=1}^5 c_i^2 I_2$. Thus $\{\varphi'_j\}_{j=1}^5$ represents a 2D regular optimal placement with $\|\varphi'_j\| = \sqrt{1/3\sum_{i=1}^5 c_i^2 - c_j^2}$ for all $j \in \{1, \ldots, 5\}$ (it can be verified $\{\|\varphi'_j\|\}_{j=1}^5$ is regular with respect to $d = 2$). Therefore, to obtain Φ, we can first construct $\{\varphi'_j\}_{j=1}^5$ using Algorithm **1** for example, and then find Φ in the orthogonal complement of the row space of Φ'. Since $\{\varphi'_i\}_{i=1}^5$ may have non-equivalent solutions, $\{\varphi_i\}_{i=1}^5$ would not be unique up to the equivalence.

5.4 *Distributed Construction*

When there are a large number of sensors, it might be inconvenient to design the optimal placement involving all sensors. The following property can be applied to construct large-scale optimal placements in a distributed manner. The 2D versions of the following result have been proposed in Bishop et al. (2010), Doğançay and Hmam (2008), Bishop and Jensfelt (2009).

Theorem 5.6 : *The union of multiple disjoint regular optimal placements in $\mathbb{R}^d$ ($d = 2$ or 3) is still a regular optimal placement in $\mathbb{R}^d$.*

Proof In $\mathbb{R}^d$, consider multiple disjoint regular optimal placements: $\{c_i, g_i\}_{i\in\mathcal{I}_k}$ with $\mathcal{I}_k$ as the index set of the kth placement ($k = 1, \dots, q$). The term disjoint as used here means that different placements share no common sensors. Define $\mathcal{I} = \bigcup_{k=1}^q \mathcal{I}_k$. If $|\cdot|$ denotes the cardinality of a set, then $|\mathcal{I}| = \sum_{k=1}^q |\mathcal{I}_k|$.

For the kth placement, since $\{c_i, g_i\}_{i\in\mathcal{I}_k}$ is regular optimal in $\mathbb{R}^d$, from Theorem 4.1 we have

$$\sum_{i\in\mathcal{I}_k} c_i^2 g_i g_i^T = \frac{1}{d}\sum_{i\in\mathcal{I}_k} c_i^2 I_d.$$

For the union placement $\{c_i, g_i\}_{i\in\mathcal{I}}$, we have

$$\begin{aligned}\sum_{j\in\mathcal{I}} c_j^2 g_j g_j^T &= \sum_{k=1}^q \sum_{i\in\mathcal{I}_k} c_i^2 g_i g_i^T \\ &= \frac{1}{d}\sum_{k=1}^q \sum_{i\in\mathcal{I}_k} c_i^2 I_d \\ &= \frac{1}{d}\sum_{j\in\mathcal{I}} c_j^2 I_d.\end{aligned}$$

By Theorem 4.1, the union placement is regular optimal in $\mathbb{R}^d$. □

Theorem 5.6 implies that a large-scale regular optimal placement can be constructed in a distributed manner: firstly divide the large-scale placement into a number of disjoint regular sub-placements, secondly construct each regular optimal sub-placement, and finally combine these optimal sub-placements together to obtain a large regular optimal placement. We call this kind of method as *distributed construction*. Because the combination of the optimal sub-placements can be arbitrary, distributed construction will lead to an infinite number of optimal placements for the large system. These optimal placements have the same FIM and $\|G\|^2$, but they are generally *non-equivalent*. Theorem 5.6 also implies that only regular placements can be possibly divided into some regular subsets.

Figure 8 gives examples of optimal placements generated by distributed construction. Suppose all sensors have the same measurement noise standard deviation. The placement with $n = 6$ in Figure 8 (a), (b) or (c) is a combination of two regular triangles with $n = 3$ (the sensors with the same color form a triangle optimal placement). The placement with $n = 8$ in Figure 8 (d) or (e) is a combination of two regular optimal placements with $n = 4$ as shown in Figure 7 (c), which is equivalent to the regular tetrahedron. Thus by Theorem 5.6 all placements in Figure 8 are regular optimal.

The distributed construction method is suitable for (but not limited to) constructing equally-weighted optimal placements. That is because a equally-weighted placement can be easily divided into some regular subsets. Suppose we have an equally-weighted placements in $\mathbb{R}^d$. Its coefficient sequence $\{c_i\}_{i=1}^n$ satisfies $c_i = c_j$ for all $i \neq j$. Then $\{c_i\}_{i=1}^n$ can be divided into a number of subsets. As long as the cardinality of each subset is no smaller than d, the subsets are all regular (refer to Example 2.2). We next present two examples to show how to divide equally-weighted placements into subsets. (i) For any integer $n \geq 4$, it is obvious that there exist nonnegative integers m_1 and m_2 such that n can be decomposed as $n = 2m_1 + 3m_2$. Thus in $\mathbb{R}^2$ we can always distributedly construct an equally-weighted optimal placement with $n \geq 4$ by using the ones with $n = 2$ or 3. (ii) For any integer $n \geq 6$, there exist nonnegative integers m_1, m_2 and m_3 such that $n = 3m_1 + 4m_2 + 5m_3$. Thus in $\mathbb{R}^3$ we can always distributedly construct an

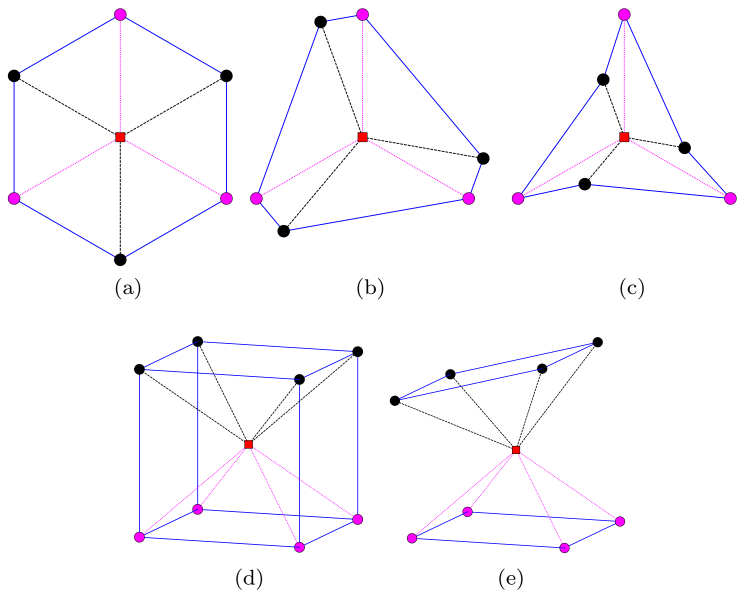


Figure 8.: Examples of distributedly constructed optimal placements. Red square: target; dots: sensors.

equally-weighted optimal placement with $n \geq 6$ by using the ones with $n = 3$, 4 or 5. Note distributed construction yields an infinite number of non-equivalent optimal placements. Hence the above two examples also imply that equally-weighted placements with $n \geq 4$ in $\mathbb{R}^2$ or $n \geq 6$ in $\mathbb{R}^3$ are not unique.

6 Numerical Verification

In order to verify our previous analysis, in this section we solve the parameter optimization problem (8) from a numerical perspective. More specifically, we employ Lyapunov approaches to design a centralized gradient control law which can numerically minimize the objective function $\|G\|^2$ given an appropriate initial condition. The control law can be applied to numerically construct generic regular and irregular optimal placements in 2D and 3D.

Assume the motion model of sensor i to be $\dot{s}_i = u_i$, where $u_i \in \mathbb{R}^d$ is the control input. Then we have $\dot{r}_i = u_i$ because $r_i = s_i - p$ and the target position estimation p is given. Let $r = [r_1^T, \dots, r_n^T]^T \in \mathbb{R}^{dn}$. Denote β as the constant lower bound of $\|G\|^2$ in Theorems 4.1 and 4.3. Then the optimal placement set is $\mathcal{E}_0 = \{r \in \mathbb{R}^{dn} : \|G\|^2 - \beta = 0\}$. Choose the Lyapunov function as $V(r) = 1/4(\|G\|^2 - \beta)$. Clearly V is positive definite with respect to $\mathcal{E}_0$. Denote $\partial V/\partial r_i$ as the Jacobian of V with respect to r_i. Then we have

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial r_i} \dot{r}_i = \sum_{i=1}^{n} \frac{c_i^2}{\|r_i\|} g_i^T G P_i \dot{r}_i,$$

where $P_i = I_d - g_i g_i^T$ is an orthogonal projection matrix satisfying $P_i^T = P_i$, $P_i^2 = P_i$, and $\text{Null}(P_i) = \text{span}\{g_i\}$. $\text{Null}(\cdot)$ denotes the null space of a matrix. Design the gradient control law as

$$\dot{r}_i = -P_i G g_i. \tag{26}$$

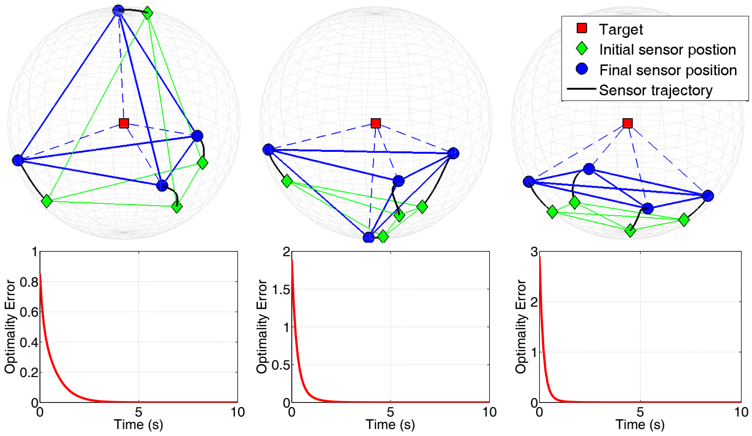
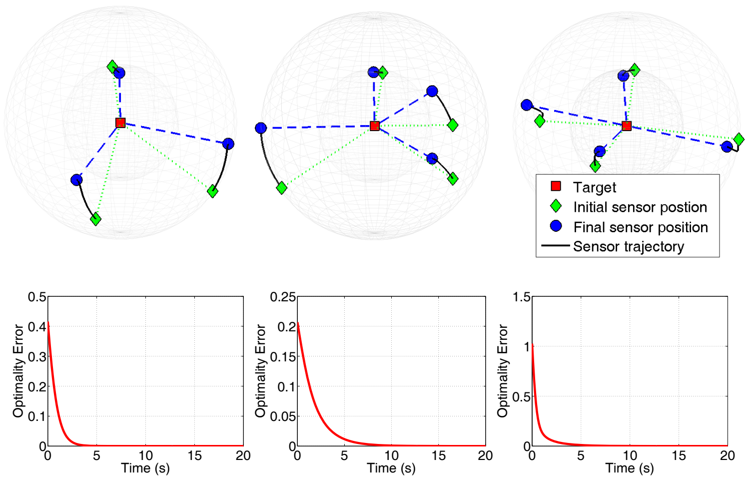


Figure 9.: Gradient control of equally-weighted (regular) placements with $n = 4$ in $\mathbb{R}^3$.



(a) $n = 3,\ k_0 = 1$ (b) $n = 4,\ k_0 = 1$ (c) $n = 4,\ k_0 = 2$

Figure 10.: Gradient control of irregular placements in $\mathbb{R}^3$.

Then

$$\dot{V} = -\sum_{i=1}^{n} \frac{c_i^2}{\|r_i\|} \|P_i G g_i\|^2 \leq 0$$

and $\dot{V} = 0$ when $P_i G g_i = 0$ for all $i \in \{1, \ldots, n\}$.

Proposition 6.1: *For any initial condition $r(0) \in \mathbb{R}^{dn}$ with $\|r_i(0)\| \neq 0$ for all $i \in \{1, \ldots, n\}$, the solution to the nonlinear r-dynamics (26) asymptotically converges to the set*

$$\mathcal{E} = \{r \in \mathcal{S} : P_i G g_i = 0, \quad i = 1, \ldots, n\},$$

where $\mathcal{S} = \{r \in \mathbb{R}^{dn} : \|r_i\| = \|r_i(0)\|,\ i = 1, \ldots, n\}$.

Proof The time derivative of $\|r_i\|$ is

$$\frac{\mathrm{d}\|r_i\|}{\mathrm{d}t} = \frac{r_i^T}{\|r_i\|} \dot{r}_i = -g_i^T P_i G g_i = 0. \tag{27}$$

The last equality uses the fact $g_i^T P_i = 0$. By (27) we have $\|r_i(t)\| \equiv \|r_i(0)\| \neq 0$. Hence $\mathcal{S}$ is a positive invariant set with respect to the r-dynamics. The set $\mathcal{S}$ consists of a group of spheres in

$\mathbb{R}^d$ centered at the origin. Thus $\mathcal{S}$ is compact. Note $\dot{V} = 0$ and $\dot{r} = 0$ for all points in $\mathcal{E}$. By the invariance principle (Khalil 2002), every solution starting in $\mathcal{S}$ asymptotically converges to $\mathcal{E}$. □

By Proposition 6.1, the r-dynamics converge either to the optimal placement set $\mathcal{E}_0$ or the set $\mathcal{E} \setminus \mathcal{E}_0$. By introducing Lagrange multipliers γ_i, $i = 1, \dots, n$, the constrained optimization problem (8) is equivalent to minimizing the Lagrangian function $L = \|G\|^2 + \sum_{i=1}^n \gamma_i (g_i^T g_i - 1)$. By calculating $\partial L / \partial g_i = 0$, we can show that $\mathcal{E}$ is the *critical point* set, which consists of not only minimizers of $\|G\|^2$ (i.e., optimal placements) but also saddle points and maximizers of $\|G\|^2$ (i.e., non-optimal placements). The sets $\mathcal{E}_0$ and $\mathcal{E}$ are equilibrium manifolds. It is noticed that nonlinear stabilization problems involving equilibrium manifolds also emerge in formation control area recently (Krick et al. 2009, Dörfler and Francis 2010, Summers et al. 2011). It is possible to conduct strict stability analysis including identifying the attractive region of $\mathcal{E}_0$ by using center manifold theory (Krick et al. 2009, Summers et al. 2011) or differential geometry (Dörfler and Francis 2010). But that will be non-trivial because the geometric structure of $\mathcal{E}_0$ is extremely complicated as shown in Dykema and Strawn (2006).

Figure 9 and Figure 10 show several optimal placements obtained by the proposed gradient control law. Due to space limitations, we only show 3D examples. The three final converged placements in Figure 9 are actually the three regular optimal ones shown in Figure 7. The three final placements in Figure 10 are the two as illustrated in Figure 2 (b) and (c). Clearly the numerical results are consistent with our previous analysis. The *optimality error* refers to the difference between $\|G\|^2$ and its lower bound given in (10) or (13). The optimality error can be used as a numerical indicator to evaluate the optimality of a placement. As shown in Figure 9 and Figure 10, the optimality errors all converge to zero.

7 Conclusions

In this paper, the optimal sensor placement problem was formulated as a parameter optimization problem. We presented a unified framework to analyze optimal placements of bearing-only, range-only, and RSS sensors. We proved the necessary and sufficient conditions for optimal placements in 2D and 3D. A number of important analytical properties of optimal placements have also been explored. The results presented in this work can be applied to analytically evaluate and construct any generic optimal placements in 2D or 3D. The proposed gradient control law not only verifies our previous analysis, but also provides us a convenient way to construct optimal placements numerically.

There are several directions for future research. First, the gradient control law proposed in this paper is a centralized control based on all-to-all communications. It is meaningful to study distributed optimization algorithms that can distributedly minimize the objective function in (8). Second, the existing work mainly considers the case of one single target. Hence another interesting research direction is to analyze the optimal sensor placements for tracking multiple targets. Third, this work focuses on homogeneous sensor networks. It is also important to analytically characterize 2D or 3D optimal placements of heterogeneous sensors in the future.

References

J. J. Benedetto and M. Fickus. Finite normalized tight frames. *Advances in Computational Mathematics*, 18:357–385, 2003.

A. N. Bishop and P. Jensfelt. An optimality analysis of sensor-target geometries for signal strength based localization. In *Proceedings of the 5th International Conference on Intelligent Sensors, Sensor Networks, and Information Processing*, pages 127–132, Melbourne, Australia, December 2009.

A. N. Bishop and M. Smith. Remarks on the cramer-rao inequality for doppler-based target parameter estimation. In *Proceedings of the 6th International Conference on Intelligent Sensors, Sensor Networks and Information Processing*, pages 199–204, Brisbane, Australia, December 2010.

A. N. Bishop, B. Fidan, B. D. O. Anderson, P. N. Pathirana, and K. Doğançay. Optimality analysis of sensor-target geometries in passive localization: Part 2 - time-of-arrival based localization. In *Proceedings of the 3rd International Conference on Intelligent Sensors, Sensor Networks and Information Processing*, pages 13–18, Melbourne, Australia, December 2007.

A. N. Bishop, B. Fidan, B. D. O. Anderson, K. Doğançay, and P. N. Pathirana. Optimality analysis of sensor-target localization geometries. *Automatica*, 46:479–492, 2010.

J. Cahill, M. Fickus, D. G. Mixon, M. J. Poteet, and N. K. Strawn. Constructing finite frames of a given spectrum and set of lengths. preprint.

P. G. Casazza and M. T. Leon. Existence and construction of finite tight frames. *Journal of Computational and Applied Mathematics*, 4(3):277–289, 2006.

P. G. Casazza, M. Fickus, J. Kovačević, M. T. Leon, and J. C. Tremain. A physical interpretation of tight frames. In *Harmonic Analysis and Applications*, Applied and Numerical Harmonic Analysis, pages 51–76. Birkh 2006.

P. G. Casazza, M. Fickus, and D. G. Mixon. Auto-tuning unit norm frames. *Applied and Computational Harmonic Analysis*, 32:1–15, 2012.

F. Dörfler and B. Francis. Geometric analysis of the formation problem for autonomous robots. *IEEE Transactions on Automatic Control*, 55(10):2379–2384, October 2010.

K. Doğançay. Online optimization of receiver trajectories for scan-based emitter localization. *IEEE Transactions on Aerospace and Electronic Systems*, 43(3):1117–1125, July 2007.

K. Doğançay and H. Hmam. Optimal angular sensor separation for AOA localization. *Signal Processing*, 88:1248–1260, 2008.

K. Dykema and N. Strawn. Manifold structure of spaces of spherical tight frames. *International Journal of Pure and Applied Mathematics*, 28:217–256, 2006.

D. Feng, L. Wang, and Y. Wang. Generation of finite tight frames by householder transformations. *Advances in Computational Mathematics*, 24:297–309, 2006.

M. Fickus. *Finite normalized tight frames and spherical equidistribution.* PhD thesis, University of Maryland, 2001.

V. K. Goyal, J. Kovačević, and J. A. Kelner. Quantized frame expansions with erasures. *Applied and Computational Harmonic Analysis*, 10(3):203–233, May 2001.

J. T. Isaacs, D. J. Klein, and J. P. Hespanha. Optimal sensor placement for time difference of arrival localization. In *Procedings of the 48th Conference on Decision and Control*, pages 7878–7884, Shanghai, China, December 2009.

D. B. Jourdan and N. Roy. Optimal sensor placement for agent localization. In *Proceedings of the IEEE/ION Position, Location, and Navigation Symposium*, pages 128–139, San Diego, USA, April 2006.

H. K. Khalil. *Nonlinear Systems, Third Edition.* Prentice Hall, 2002.

J. Kovačević and A. Chebira. Life beyond bases: the advent of frames (part I). *IEEE Signal Processing Magazine*, 24(4):86–104, July 2007a.

J. Kovačević and A. Chebira. Life beyond bases: the advent of frames (part II). *IEEE Signal Processing Magazine*, 24(5):115–125, September 2007b.

L. Krick, M. E. Broucke, and B. A. Francis. Stabilization of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, 82(3):423–439, 2009.

S. Martínez and F. Bullo. Optimal sensor placement and motion coordination for target tracking. *Automatica*, 42:661–668, 2006.

W. Meng, L. Xie, and W. Xiao. Optimality analysis of sensor-source geometries in heterogeneous sensor networks. *IEEE Transactions on Wireless Communications.* to appear.

B. M. Miller and E. Y. Rubinovich. *Impulsive Control in Continuous and Discrete-Continuous Systems.* Kluwer Academic/Plenum Publishres, New York, 2003.

D. Moreno-Salinas, A. M. Pascoal, and J. Aranda. Optimal sensor placement for underwater positioning with uncertainty in the target location. In *Proceedings of the 2011 IEEE International Conference on Robotics and Automation*, pages 2308–2314, Shanghai, China, May 2011.

Y. Oshman and P. Davidson. Optimization of observer trajectories for bearings-only target localization. *IEEE Transactions on Aerospace and Electronic Systems*, 35(3):892–902, July 1999.

J. Ousingsawat and M. E. Campbell. Optimal cooperative reconnaissance using multiple vehicles. *Journal of Guidance, Control, and Dynamics*, 30(1):122–132, January–February 2007.

F. Pukelsheim. *Optimal Design of Experiments*. John Wiley, 1993.

A. J. Sinclair, R. J. Prazenica, and D. E. Jeffcoat. Optimal and feedback path planning for cooperative attack. *Journal of Guidance, Control, and Dynamics*, 31(6):1708–1715, November–December 2008.

T. H. Summers, C. Yu, S. Dasgupta, and B. D. O. Anderson. Control of minimally persistent leader-remote-follower and coleader formations in the plane. *IEEE Transactions on Automatic Control*, 56(12):2778–2792, December 2011.

C. Yang, L. Kaplan, E. Blasch, and M. Bakich. Optimal placement of heterogeneous sensors in target tracking. In *Proceedings of the 14th International Conference on Information Fusion*, pages 1–8, Chicago, USA, July 2011.

H. Zhang. Two-dimensional optimal sensor placement. *IEEE Transactions on Systems, Man, and Cybernetics*, 25(5):781–792, May 1995.

S. Zhao, B. M. Chen, and T. H. Lee. Optimal placement of bearing-only sensors for target localization. In *Procedings of the 2012 Amarican Control Conference*, pages 5108–5113, Montreal, Canada, June 2012.