Optimal Placement of Sensor Networks for Target Localization and Tracking

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1. What the problem is intuitively

2. What the problem is mathematically

3. How to solve the problem

4. Other Interesting Properties

5. Conclusions
Outline

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Problem Description

Cooperative target localization:

- Objective: localize the target
- Measurement: partial information of the target such as bearing or range
- Cooperative localization: sensors must collaborate to localize the target
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  - **Range-only**: Martínez and Bullo [2006], Jourdan and Roy [2006], Bishop et al. [2010], Morales and Kassas [2015]
  
  - **Received-Signal-Strength (RSS)**: Bishop and Jensfelt [2009]
  
  - **Other Types**: ...
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- **Space Dimension**
  - **2D Space**: Zhang [1995], Martínez and Bullo [2006], Jourdan and Roy [2006], Bishop et al. [2007], Doğançay [2007], Doğançay and Hmam [2008], Bishop and Jensfelt [2009], Isaacs et al. [2009], Bishop et al. [2010], Morales and Kassas [2015]
  - **3D Space**: Moreno-Salinas et al. [2011]

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- Sensor type: bearing-only, range-only, and RSS-based sensors
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The measurement model for sensor $i$ is

- **Bearing-only sensor:** $h(s_i - p) = s_i - p$
- **Range-only sensor:** $h(s_i - p) = \|s_i - p\|$
- **RSS-based sensor:** $h(s_i - p) = \ln \|s_i - p\|$

$g_i$ is the measurement vector from sensor $i$ to the target $p$. $s_i$ is the sensor location and $p$ is the target location.
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Optimality Metrics

Fisher Information Matrix (FIM):

\[ F = \sum_{i=1}^{n} \left[ \frac{\partial h}{\partial p} \right]^T \Sigma_i^{-1} \frac{\partial h}{\partial p} \]

- \( \Sigma_i \): how accurate the measurement is
- \( \frac{\partial h}{\partial p} \): how much useful information the measurement has
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- **T-Optimality:** maximize $\text{tr } F = \sum_{i=1}^{d} \lambda_i$
- **A-Optimality:** minimize $\text{tr } F^{-1} = \sum_{i=1}^{d} 1/\lambda_i$
- **D-Optimality:** maximize $\text{det } F = \prod_{i=1}^{d} \lambda_i$

**Interpretation:**
- Maximize the information obtained by the sensors
- Minimize the volume of the uncertainty ellipsoid

**Limitation:** not applicable to 3-D cases
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- 2-D case: $$\| F - \bar{\lambda}I_2 \|^2 = -2 \det F + 2\bar{\lambda}^2 \text{ (equivalent)}$$
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  - **case 2:** $\det F < \bar{\lambda}^3$ and $\| F - \bar{\lambda} I_d \| > 0$ (not equivalent)
Problem Statement

For bearing-only, range-only, and RSS sensors

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\min \| F - \bar{\lambda} I_d \|^2 \iff \min \| G \|^2
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where

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G = \sum_{i=1}^{n} c_i^2 g_i g_i^T
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#### Definition (Irregularity)

- The weights for $n$ sensors: $c_1 \geq c_2 \geq \cdots \geq c_n > 0$
- Dimension: $d = 2, 3$

Denote $k_0$ as the smallest nonnegative integer $k$ for which

$$c_{k+1}^2 \leq \frac{1}{d - k} \sum_{i=k+1}^{n} c_i^2.$$ 

The integer $k_0$ is called the irregularity of $\{c_i\}_{i=1}^{n}$ with respect to $d$. 

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Regular VS Irregular

Example (Regular)
\[
\{c_i\}_{i=1}^4 = \{1, 1, 1, 1\} \quad \& \quad d = 3
\]

• \(k = 0\):
  \(\sum_{i=1}^{n} c_{2i} = 1\)

Irregularity = 0

Example (Irregular)
\[
\{c_i\}_{i=1}^4 = \{10, 1, 1, 1\} \quad \& \quad d = 3
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• \(k = 0\):
  \(\sum_{i=1}^{n} c_{2i} = 10\)>

Irregularity = 1

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Irregularity = 2

Intuition
A sequence is irregular when certain element is much larger than the others.
Regular VS Irregular

Example (Regular)

\[ \{c_i\}_{i=1}^4 = \{1, 1, 1, 1\} \] & \(d = 3:\)

- \(k = 0: 1 = c_1^2 \leq \frac{1}{d} \sum_{i=1}^n c_i^2 = 1.3\)

Irregularity=0
Regular VS Irregular

Example (Regular)

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\( \{c_i\}_{i=1}^{4} = \{10, 1, 1, 1\} \) & \( d = 3 \):

- \( k = 0 \): \( 100 = c_1^2 > \frac{1}{d} \sum_{i=1}^{n} c_i^2 = 34.3 \)
- \( k = 1 \): \( 1 = c_2^2 \leq \frac{1}{d-1} \sum_{i=2}^{n} c_i^2 = 1.5 \)

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- \(k = 0\): \(100 = c_1^2 > \frac{1}{d} \sum_{i=1}^{n} c_i^2 = 67.3\)
- \(k = 1\): \(100 = c_2^2 > \frac{1}{d-1} \sum_{i=2}^{n} c_i^2 = 51\)
- \(k = 2\): \(1 = c_2^2 \leq \frac{1}{d-2} \sum_{i=3}^{n} c_i^2 = 2\)

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Intuition

A sequence is irregular when certain element is much larger than the others.
Necessary and Sufficient Condition for Optimal Placement

**Theorem (Regular optimal placement)**

In $\mathbb{R}^d$ with $d = 2$ or $3$, if the positive coefficient sequence $\{c_i\}_{i=1}^n$ is regular, then the objective function $\|G\|^2$ satisfies

$$\|G\|^2 \geq \frac{1}{d} \left( \sum_{i=1}^n c_i^2 \right)^2.$$  

The lower bound of $\|G\|^2$ is achieved if and only if

$$\sum_{i=1}^n c_i^2 g_i g_i^T = \frac{1}{d} \sum_{i=1}^n c_i^2 I_d.$$
Necessary and Sufficient Condition for Optimal Placement

**Theorem (Regular optimal placement)**

In $\mathbb{R}^d$ with $d = 2$ or $3$, if the positive coefficient sequence $\{c_i\}_{i=1}^n$ is regular, then the objective function $\|G\|^2$ satisfies

$$\|G\|^2 \geq \frac{1}{d} \left( \sum_{i=1}^n c_i^2 \right)^2.$$

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**Theorem (Irregular optimal placement)**

In $\mathbb{R}^d$ with $d = 2$ or $3$, if the positive coefficient sequence $\{c_i\}_{i=1}^n$ is irregular with irregularity as $k_0 \geq 1$, without loss of generality $\{c_i\}_{i=1}^n$ can be assumed to be a non-increasing sequence, and then the objective function $\|G\|^2$ satisfies

$$\|G\|^2 \geq \sum_{i=1}^{k_0} c_i^4 + \frac{1}{d - k_0} \left( \sum_{i=k_0+1}^n c_i^2 \right)^2.$$

The lower bound of $\|G\|^2$ is achieved if and only if

$$\{g_i\}_{i=1}^n = \{g_i\}_{i=1}^{k_0} \cup \{g_i\}_{i=k_0+1}^n,$$

where $\{g_i\}_{i=1}^{k_0}$ is an orthogonal set, and $\{g_i\}_{i=k_0+1}^n$ forms a regular optimal placement in the $(d - k_0)$-dimensional orthogonal complement of $\{g_i\}_{i=1}^{k_0}$.
Necessary and Sufficient Condition for Optimal Placement

Theorem (Regular optimal placement)

In $\mathbb{R}^d$ with $d = 2$ or $3$, if the positive coefficient sequence $\{c_i\}_{i=1}^n$ is regular, then the objective function $\|G\|_2^2$ satisfies

$$\|G\|_2^2 \geq \frac{1}{d} \left( \sum_{i=1}^{n} c_i^2 \right)^2.$$

The lower bound of $\|G\|_2^2$ is achieved if and only if

$$\sum_{i=1}^{n} c_i^2 g_i g_i^T = \frac{1}{d} \sum_{i=1}^{n} c_i^2 I_d.$$

Theorem (Irregular optimal placement)

In $\mathbb{R}^d$ with $d = 2$ or $3$, if the positive coefficient sequence $\{c_i\}_{i=1}^n$ is irregular with irregularity as $k_0 \geq 1$, without loss of generality $\{c_i\}_{i=1}^n$ can be assumed to be a non-increasing sequence, and then the objective function $\|G\|_2^2$ satisfies

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Observation

An irregular optimal sensor placement problem can be converted to a regular optimal sensor placement in a lower dimensional space.
Necessary and Sufficient Condition for Optimal Placement

2-D example:

\{c_1, c_2, c_3\} = \{1, 1, 1\}
Necessary and Sufficient Condition for Optimal Placement

2-D example:

\{c_1, c_2, c_3\} = \{1, 1, 1\} \quad \{c_1, c_2, c_3\} = \{10, 1, 1\}
Necessary and Sufficient Condition for Optimal Placement

2-D example:

\{c_1, c_2, c_3\} = \{1,1,1\}

\{c_1, c_2, c_3\} = \{10,1,1\}

3-D example:

\{c_1, c_2, c_3, c_4\} = \{1,1,1,1\}
Necessary and Sufficient Condition for Optimal Placement

2-D example:

\{c_1, c_2, c_3\} = \{1, 1, 1\} \quad \{c_1, c_2, c_3\} = \{10, 1, 1\}

3-D example:

\{c_1, c_2, c_3, c_4\} = \{1, 1, 1, 1\} \quad \{c_1, c_2, c_3, c_4\} = \{10, 1, 1, 1\}
Necessary and Sufficient Condition for Optimal Placement

2-D example:

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\[ \{c_1, c_2, c_3\} = \{10,1,1\} \]

3-D example:

\[ \{c_1, c_2, c_3, c_4\} = \{1,1,1,1\} \]

\[ \{c_1, c_2, c_3, c_4\} = \{10,1,1,1\} \]

\[ \{c_1, c_2, c_3, c_4\} = \{10,10,1,1\} \]
Outline

1. What the problem is intuitively
2. What the problem is mathematically
3. How to solve the problem
4. Other Interesting Properties
5. Conclusions
Equally-weighted sensor networks:

Figure: Examples of 2D equally-weighted optimal placements
Other Interesting Properties

Equally-weighted sensor networks:

Figure: Examples of 2D equally-weighted optimal placements

Figure: Examples of 3D equally-weighted optimal placements
Theorem (Distributed construction)

The union of multiple disjoint regular optimal placements in $\mathbb{R}^d$ ($d = 2$ or $3$) is still a regular optimal placement in $\mathbb{R}^d$.
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The union of multiple disjoint regular optimal placements in $\mathbb{R}^d$ ($d = 2$ or $3$) is still a regular optimal placement in $\mathbb{R}^d$. 
\[ V = \|G\|^2/4. \] Gradient descent control:

\[ \dot{r}_i = -\left( \frac{\partial V}{\partial r_i} \right)^T = -P_i G g_i. \]
Autonomous Optimal Sensor Deployment

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Regular placements:
$$V = \|G\|^2/4.$$ Gradient descent control:

$$\dot{r}_i = -\left(\frac{\partial V}{\partial r_i}\right)^T = -P_i G g_i.$$  

Regular placements:

Irregular placements:

(g) $n = 3$, $k_0 = 1$

(h) $n = 4$, $k_0 = 1$

(i) $n = 4$, $k_0 = 2$
(a) Trajectory
Autonomous Optimal Sensor Deployment

(a) Trajectory

(b) Estimation error
1. What the problem is intuitively
2. What the problem is mathematically
3. How to solve the problem
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Conclusions

Contributions:

1. Optimality metrics:
   \[ \max \operatorname{det} F \Rightarrow \min \| F - \bar{\lambda} I \|_2 \Rightarrow \min \| G \|_2 \]

2. Necessary and sufficient conditions:
   - Regular
   - Irregular

3. Other properties:
   - Equally-weighted sensors
   - Distributed construction

Future work:

1. Control strategy
2. Multiple targets or target area
Conclusions

Contributions:

1. Optimality metrics

\[
\max \det F \implies \min \|F - \bar{\lambda}I_d\|^2 \implies \min \|G\|^2
\]
Conclusions

Contributions:

1. Optimality metrics

\[
\text{max det } F \implies \min \| F - \lambda I_d \|^2 \implies \min \| G \|^2
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The End

Q & A


J. T. Isaacs, D. J. Klein, and J. P. Hespanha. Optimal sensor placement for


