Globally Convergent Distributed Network Localization Using Locally Measured Bearings

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***Abstract*—This paper studies the problem of bearing-based network localization, which aims to estimate the absolute positions of the nodes in a network by using the inter-node bearings measured in each node's local reference frame and the absolute positions of a small number of nodes called anchors. In the first part of the paper, we propose a continuous-time localization algorithm, which consists of coupled orientation and position estimation procedures. Compared to the existing works, the proposed algorithm has a concise form and guarantees global estimation convergence. In the second part of the paper, we study the discrete-time case which is still an open problem till now. We fill this gap by proposing a discrete-time localization algorithm to globally localize three-dimensional networks using locally measured bearings. The discrete-time algorithm does not require designing sufficiently small step sizes to ensure convergence. Numerical simulation is presented to verify the proposed algorithms.**

***Index Terms*—Bearing measurements, network localization, orientation estimation.**

I. Introduction

BEARING-BASED network localization studies how to localize a network of sensing nodes where each node can only measure the relative bearings to their nearest neighbors. This problem has received increasing research attention recently due to the rapid development of bearing-only sensors, such as optical cameras [1] and sensor arrays [2]. Compared to the conventional approach of distance-based network localization, where each node can measure the distances to their neighbors, the advantage of the bearing-based approach is that it can be formulated as a linear dynamical system whose global stability can be easily analyzed [3]. However, one disadvantage of the bearing-based approach is that each bearing, which can be represented by a unit vector, must be expressed in a global or local reference frame. The problem of bearing-based network localization in the presence of global reference frames has been studied extensively [4]–[10]. However, global reference frames may not be accurately measured in many environments, such as indoors or urban canyons. The case without a global reference frame known to each node deserves more research attention, and is also the focus of our work in this paper.

A general approach to utilize measurements obtained in local reference frames is orientation estimation. In particular, each node could estimate its absolute orientation with respect to a global reference frame using inter-neighbor relative orientation measurements. Then, the estimated absolute orientations can be used to convert the local measurements into the global reference frame. This approach has been applied in formation control and network localization problems [11]–[13]. Very recently, this approach has also been applied to solve the problem of bearing-based network localization in [14]. However, the algorithm in [14] requires a Gram–Schmidt orthogonalization procedure at each time instance. This orthogonalization procedure could not ensure global convergence because singular matrices could not be converted to rotation matrices by orthogonalization.

The problem of estimating fixed orientations should be distinguished from the problem of orientation control or synchronization [15]–[20]. The latter problem requires that the orientation matrix obtained at every time instance be constrained to be a rotation matrix. As a comparison, the estimation of fixed orientations is not subject to this constraint, although the estimated orientations should eventually converge to the true orientations which are rotation matrices. Therefore, the orthogonalization procedure in the algorithms in [13] and [14] is redundant when the orientations to be estimated are fixed. After dropping the orthogonalization procedure, the orientation estimation problem could be simply formulated as a linear time-invariant system as shown in this paper.

The contributions of this paper are twofold. The first contribution is to propose a continuous-time network localization algorithm merely using local bearing and relative orientation measurements. This algorithm consists of coupled orientation and position estimation procedures. Since no orthogonalization is needed, the algorithm has a concise form and, more importantly, guarantees global convergence as opposed to [14]. The second contribution is to solve the problem of discrete-time bearing-based network localization. This problem has not been completely solved even in the presence of global reference frames, although the work in [21] has solved the two-dimensional case.

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We propose a new discrete-time bearing-based localization algorithm that can localize three-dimensional networks in the presence of a global reference frame known to each node. We further propose algorithms to solve the case without global reference frames. The proposed discrete-time algorithms are globally convergent and do not require designing sufficiently small step sizes to ensure system convergence.

The paper is organized as follows. Section II gives necessary preliminaries and the statement of the problem to be solved in this paper. The continuous-time and discrete-time network localization problems are, respectively, studied in Sections III and IV. Simulation results are given in Section V and conclusions are drawn in Section VI.

II. Preliminaries and Problem Statement

A. Notations for Network Localization

Consider n stationary sensing nodes in $\mathbb{R}^d$ ($n \geq 2$ and $d = 2, 3$). Suppose Σ^0 is a fixed global reference frame. Let $p_i \in \mathbb{R}^d$ be the true position of node $i \in \{1, \ldots, n\}$ expressed in Σ^0 and $p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{dn}$. Suppose Σ^i is a *fixed* body reference frame for node i. The origin of Σ^i is at p_i. Let $R_i \in \mathbb{R}^{d\times d}$ satisfying $R_i^T R_i = I_d$ and $\det R_i = 1$ be the rotation matrix from Σ^i to Σ^0. The orientation of node i is represented by R_i.

The interaction among the nodes is described by a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which consists of a vertex set $\mathcal{V} = \{1, \ldots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The edge $(i, j) \in \mathcal{E}$ indicates that node i can measure the relative bearing of node j and, in the meantime, can receive necessary information from node j via wireless communication. Then, node j is called a neighbor of i and the set of neighbors of node i is denoted as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. This paper only consider *connected* and *undirected* graphs where $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. A network, denoted as $(\mathcal{G}, p)$, is $\mathcal{G}$ with its vertex i mapped to p_i for all $i \in \mathcal{V}$.

Suppose the first n_a nodes, called anchors, know their true positions and orientations. The rest $n_f = n - n_a$ nodes, called followers, do not know their own positions or orientations. Let $\mathcal{V}_a = \{1, \ldots, n_a\}$ and $\mathcal{V}_f = \mathcal{V} \backslash \mathcal{V}_a$ be the sets of anchors and followers, respectively. Let $p_a = [p_1^T, \ldots, p_{n_a}^T]^T$ and $p_f = [p_{n_a+1}^T, \ldots, p_n^T]^T$. Then $p = [p_a^T, p_f^T]^T$.

If $(i, j) \in \mathcal{E}$, then

$$g_{ij} := \frac{p_j - p_i}{\|p_j - p_i\|}$$

is the unit vector representing the relative bearing of p_j with respect to p_i. Note that $g_{ij} = -g_{ji}$. In this paper, $\|\cdot\|$ denotes the Euclidean norm of a vector or the spectral norm of a matrix. Let $g_{ij}^{(i)}$ be the local version of bearing g_{ij} measured by node i in Σ^i. Specifically

$$g_{ij} = R_i g_{ij}^{(i)}.$$

For a vector $x \in \mathbb{R}^d$, define

$$P_x := I_d - xx^T$$

where $I_d \in \mathbb{R}^{d\times d}$ is the identity matrix. If x is a unit vector with $\|x\| = 1$, it can be verified that P_x is positive semidefinite and $\text{Null}(P_x) = \text{span}\{x\}$. Geometrically, P_x is an orthogonal projection matrix that can project any vector onto the orthogonal compliment of x. This orthogonal projection matrix is widely used in bearing-based control and estimation problems because it is able to describe parallel bearing vectors in arbitrary dimensions [3], [10], [22]. It must be noted that, if x is not a unit vector (i.e., $\|x\| \neq 1$), P_x may not be positive semidefinite anymore.

B. Problem Statement

Suppose $\hat{p}_i$ and $\hat{R}_i$ are the estimates of position p_i and orientation R_i for $i \in \mathcal{V}$, respectively. If $i \in \mathcal{V}_a$, then $\hat{p}_i = p_i$ and $\hat{R}_i = R_i$. The problem to be solved in this paper is formally stated as below.

Problem 1 (Bearing-Based Network Localization): For network $(\mathcal{G}, p)$, design a distributed algorithm to estimate the position p_i and orientation R_i of follower $i \in \mathcal{V}_f$ merely using the local bearing measurements $\{g_{ij}^{(i)}\}_{j\in\mathcal{N}_i}$ and relative orientations $\{R_i R_j^T\}_{j\in\mathcal{N}_i}$ such that $\hat{p}_i$ and $\hat{R}_i$ converge to p_i and R_i, respectively.

In order to solve Problem 1, two questions must be answered. The first is whether the network can be possibly localized. The second is, if the network is localizable, how to localize it in a distributed manner. The answer to the first question is bearing localizability [10], the preliminaries of which are given in the next subsection. The second question is the focus of the rest of the paper.

In Problem 1, each node is supposed to be able to measure the local bearings and the relative orientations of its neighbors. While bearings can be measured by bearing-only sensors, it may be difficult to directly measure relative orientations in practice. Fortunately, relative orientations may be calculated using local bearing measurements. In particular, consider a pair of neighboring nodes i and j whose local bearing measurements are $g_{ij}^{(i)}$ and $g_{ji}^{(j)}$. Suppose the two nodes share their local bearings with each other via wireless communication. Since $g_{ij} = R_i g_{ij}^{(i)}$, $g_{ji} = R_j g_{ji}^{(j)}$, and $g_{ij} = -g_{ji}$, we have $R_i g_{ij}^{(i)} = -R_j g_{ji}^{(j)}$ and hence

$$(R_j^T R_i) g_{ij}^{(i)} = -g_{ji}^{(j)}. \quad (1)$$

The relative orientation $R_j^T R_i$ can be calculated from $g_{ij}^{(i)}$ and $g_{ji}^{(j)}$ based on (1) as described below. 1) For networks in two dimensions, $R_j^T R_i$ is a rotation matrix parameterized by the angle between $g_{ij}^{(i)}$ and $-g_{ji}^{(j)}$. 2) For networks in three dimensions, if one axis of the body frames of all the nodes are aligned, this is equivalent to the two-dimensional case. The alignment of one axis of the body frames could be realized by, for example, using gravity sensors which provide a three-dimensional vector indicating the direction of the gravity force. 3) For general networks without such an alignment, there would exist an infinite number of $R_j^T R_i$ satisfying (1). In this case, the calculation of the relative orientations requires more sophisticated algorithms as shown in [6]. Note that the results in [6] are only applicable to some special three-dimensional networks. It still remains an open problem to obtain relative orientations from local bearings in general networks. This problem is out of the scope of this paper and will be studied in the future. It, however, should be noted that the results presented in this paper are applicable to general networks in three dimensions as long as the relative orientations could be obtained in any way.

Finally, it is worth noting that $\{R_i R_j^T\}_{j\in\mathcal{N}_i}$ and $\{R_j^T R_i\}_{j\in\mathcal{N}_i}$ are different relative orientation measurements because they do not imply each other since R_i and R_j^T may not commute. However, either of the measurements can be applied to orientation estimation. In particular, the algorithms proposed in this paper rely on the measurements of $\{R_i R_j^T\}_{j\in\mathcal{N}_i}$. Otherwise, if the measurements of $\{R_j^T R_i\}_{j\in\mathcal{N}_i}$ are available, we show later that the proposed algorithms can be easily modified to incorporate these measurements.

C. Preliminaries to Bearing Localizability

A network that can be localized using bearing measurements must satisfy certain architectural requirements, which are called bearing localizability conditions [10]. Preliminaries to bearing localizability are summarized as below.

Definition 1 (Bearing Localizability [10]): The network $(\mathcal{G}, p)$ is called bearing localizable if the value of p can be uniquely determined by the interneighbor bearings $\{g_{ij}\}_{(i,j)\in\mathcal{E}}$ and the positions of the anchors $\{p_i\}_{i\in\mathcal{V}_a}$.

The necessary and sufficient condition of bearing localizability can be described by the following matrix. In particular, define $\mathcal{B} \in \mathbb{R}^{dn\times dn}$ with the ijth block of submatrix as

$$[\mathcal{B}]_{ij} = \begin{cases} \mathbf{0}_{\mathbf{d}\times\mathbf{d}}, & i\neq j,\ (i,j)\notin\mathcal{E} \\ -P_{g_{ij}}, & i\neq j,\ (i,j)\in\mathcal{E} \\ \sum_{k\in\mathcal{N}_i} P_{g_{ik}}, & i=j,\ i\in\mathcal{V} \end{cases}.$$

The matrix $\mathcal{B}$ is a matrix-weighted graph Laplacian matrix. It is called the *bearing Laplacian* since it characterizes both the underlying graph and the bearings of the network. The bearing Laplacian matrix plays important roles in bearing-based control and estimation problems [10], [23]. According to the partition of anchors and followers, partition $\mathcal{B}$ as

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{aa} & \mathcal{B}_{af} \\ \mathcal{B}_{fa} & \mathcal{B}_{ff} \end{bmatrix}$$

where $\mathcal{B}_{ff} \in \mathbb{R}^{dn_f\times dn_f}$. Since the graph is undirected, it can be easily shown that $\mathcal{B}$ is symmetric positive semi definite and so is $\mathcal{B}_{ff}$. The necessary and sufficient condition of bearing localizability can be characterized by the nonsingularity of $\mathcal{B}_{ff}$ as shown below.

Lemma 1 (Condition for Bearing Localizability [10]): The network $(\mathcal{G}, p)$ is bearing localizable if and only if $\mathcal{B}_{ff}$ is nonsingular and hence positive definite. When $\mathcal{B}_{ff}$ is nonsingular, the positions of the followers can be expressed as $p_f = -\mathcal{B}_{ff}^{-1}\mathcal{B}_{fa}p_a$.

This paper only considers bearing localizable networks. Otherwise, it is impossible to localize all the followers.

In order to ensure bearing localizability, there must exist sufficient and appropriately selected anchors. It is worth noting that at least two anchors are required to ensure bearing localizability. Details on anchor selection and examples of bearing localizable networks can be found in [10].

III. Continuous-Time Localization Algorithms

This section proposes continuous-time algorithms to solve Problem 1. In particular, we first present an orientation estimation algorithm and then apply it to solve the network localization problem.

A. Continuous-Time Orientation Estimation

Let $\hat{R}_i(t)$ be the estimate of R_i at time $t \geq 0$. For $i \in \mathcal{V}_a$, $\hat{R}_i(t) = R_i$ for all t. For $i \in \mathcal{V}_f$, the proposed orientation estimation algorithm is

$$\dot{\hat{R}}_i(t) = -\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{R}_i(t) - R_i R_j^T \hat{R}_j(t)) \quad i\in\mathcal{V}_f \qquad (2)$$

where a_{ij} is a constant positive weight for edge $(i,j)\in\mathcal{E}$. The measurements required by (2) are the relative orientations $\{R_i R_j^T\}_{j\in\mathcal{N}_i}$.

The algorithm in (2) is similar to those in [13], [14], but it does not require the orthogonalization procedure. As a result, $\hat{R}_i(t)$ in (2) may *not* be a rotation matrix any more. In fact, it is unnecessary to require $\hat{R}_i(t)$ to be a rotation matrix for every t, although $\hat{R}_i(t)$ is expected to converge to R_i, which is a rotation matrix, as $t\to\infty$. If $\hat{R}_i(t)$ is not constrained as a rotation matrix, (2) is a linear matrix differential equation that can be decomposed into d vector equations by considering the d columns of $\hat{R}_i$ and proved to be globally convergent.

Theorem 1 (Orientation Estimation Convergence): If $(\mathcal{G}, p)$ is connected and undirected, given arbitrary initial estimates $\{\hat{R}_i(0)\}_{i\in\mathcal{V}_f}$, $\hat{R}_i(t)$ converges to R_i globally and exponentially fast for all $i\in\mathcal{V}_f$ by algorithm (2).

Proof: Multiplying R_i^T on both sides of (2) gives

$$R_i^T\dot{\hat{R}}_i = -\sum_{j\in\mathcal{N}_i} a_{ij}(R_i^T\hat{R}_i - R_j^T\hat{R}_j) \quad i\in\mathcal{V}_f. \qquad (3)$$

Denote

$$X_i = R_i^T\hat{R}_i, \quad i\in\mathcal{V}.$$

Let $x_{i,k}\in\mathbb{R}^d$ be the kth column of X_i where $k=1,\ldots,d$. If $i\in\mathcal{V}_a$, $X_i = I_d$ and hence $x_{i,k} = e_k$ where e_k is the kth column of the identity matrix. Substituting X_i into (3) gives

$$\dot{x}_{i,k} = -\sum_{j\in\mathcal{N}_i} a_{ij}(x_{i,k} - x_{j,k}) \quad i\in\mathcal{V}_f, k=1,\ldots,d. \qquad (4)$$

For each k, (4) is the same as the consensus algorithms with multiple leaders (i.e., containment control algorithms) [24]–[26]. Since the interactions among the followers are undirected, the underlying graph of the network has a united spanning tree.[1] As a result, $x_{i,k}$ for $i\in\mathcal{V}_f$ converges to the convex hull spanned by $x_{i,k}$ for all $i\in\mathcal{V}_a$. Since $x_{i,k}$ are equal to e_k for all $i\in\mathcal{V}_a$, it follows that $x_{i,k}$ converges to e_k for $i\in\mathcal{V}_f$. As a result,

[1] A directed graph with multiple leaders has a united directed spanning tree if, for each follower, there exists at least one leader that has a directed path to the follower [26, pp. 110].

$X_i = R_i^T \hat{R}_i$ converges to I_d and consequently $\hat{R}_i$ converges to R_i for all $i \in \mathcal{V}_f$. Since the error dynamical system is linear, the convergence is exponentially fast [27, Corollary 4.3]. ■

Two remarks on algorithm (2) are given below. 1) While the underlying sensing graph is assumed to be undirected and localizable, this graphical condition could be relaxed to having a united spanning tree as can be seen from (4). For the sake of consistency, we simply consider undirected and connected graphs in this paper. 2) Algorithm (2) requires that node i must know the relative orientations $\{R_i R_j^T\}_{(i,j)\in\mathcal{E}}$. If different measurements of $\{R_j^T R_i\}_{(i,j)\in\mathcal{E}}$ are available, we may use a slightly different algorithm

$$\dot{\hat{R}}_i(t) = -\sum_{j\in\mathcal{N}_i} a_{ij}(\hat{R}_i(t) - \hat{R}_j(t)R_j^T R_i) \quad i \in \mathcal{V}_f$$

to estimate R_i. The global stability of this algorithm can be similarly proved by multiplying R_i^T on both sides of the equation.

B. Continuous-Time Network Localization

Now we propose a distributed localization algorithm to solve Problem 1. For $i \in \mathcal{V}_a$, then $\hat{p}_i(t) = p_i$ for all t. For $i \in \mathcal{V}_f$, the proposed localization algorithm is

$$\dot{\hat{p}}_i(t) = -\sum_{j\in\mathcal{N}_i} P_{\hat{g}_{ij}(t)}(\hat{p}_i(t) - \hat{p}_j(t)) \quad i \in \mathcal{V}_f \tag{5}$$

where

$$P_{\hat{g}_{ij}(t)} = I_d - \hat{g}_{ij}(t)\hat{g}_{ij}^T(t), \quad \hat{g}_{ij}(t) = \hat{R}_i(t)g_{ij}^{(i)}.$$

The orientation estimate $\hat{R}_i(t)$ is governed by (2).

Since $\hat{R}_i(t)$ in (2) may not be a rotation matrix, $\hat{g}_{ij}$ may not be a unit vector and hence $P_{\hat{g}_{ij}(t)}$ may not be positive definite. This is the main technical challenge to analyze the system convergence.

Theorem 2 (Network Localization Convergence): If $(\mathcal{G}, p)$ is bearing localizable, given arbitrary initial estimates $\{\hat{p}_i(0)\}_{i\in\mathcal{V}_f}$ and $\{\hat{R}_i(0)\}_{i\in\mathcal{V}_f}$, $\hat{p}_i(t)$ converges to p_i globally asymptotically for all $i \in \mathcal{V}_f$ under the action of (5) and (2).

Proof: Note that (5) and (2) form a cascade nonlinear system. Its stability can be analyzed by the input-to-state-stability and Lyapunov approaches. The proof consists of three parts.

Part 1. Matrix form: The matrix-vector form of (5) is

$$\dot{\hat{p}}_f = -\hat{\mathcal{B}}_{ff}\hat{p}_f - \hat{B}_{fa}p_a \tag{6}$$

where $\hat{\mathcal{B}}_{ff}$ and $\hat{\mathcal{B}}_{fa}$ are obtained by replacing g_{ij} by $\hat{g}_{ij}$ in $\mathcal{B}_{ff}$ and $\mathcal{B}_{fa}$, respectively. Define

$$\Delta_{ff} := \hat{\mathcal{B}}_{ff} - \mathcal{B}_{ff}$$
$$\Delta_{fa} := \hat{\mathcal{B}}_{fa} - \mathcal{B}_{fa}.$$

Hence, (6) can be rewritten as

$$\begin{aligned}\dot{\hat{p}}_f &= -(\mathcal{B}_{ff} + \Delta_{ff})\hat{p}_f - (\mathcal{B}_{fa} + \Delta_{fa})p_a \\ &= -\mathcal{B}_{ff}\hat{p}_f - \mathcal{B}_{fa}p_a - (\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a) \\ &= -\mathcal{B}_{ff}(\hat{p}_f - p_f) - (\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a)\end{aligned} \tag{7}$$

where the last equality is due to $p_f = -\mathcal{B}_{ff}^{-1}\mathcal{B}_{fa}p_a$ as shown in Lemma 1. The following proof aims showing that $\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a$ converges to zero. To do that, it is necessary to show the boundedness of $\hat{p}_f$ first.

Part 2. Boundedness: Consider the Lyapunov function

$$V = \frac{1}{2}\|\hat{p}_f - p_f\|^2.$$

Let $\lambda_{\min}$ be the smallest eigenvalue of $\mathcal{B}_{ff}$. Since $\mathcal{B}_{ff}$ is positive definite, we have $\lambda_{\min} > 0$. Then,

$$\begin{aligned}\dot{V} &= (\hat{p}_f - p_f)^T\dot{\hat{p}}_f \\ &= (\hat{p}_f - p_f)^T\left(-\mathcal{B}_{ff}(\hat{p}_f - p_f) - (\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a)\right) \\ &= -(\hat{p}_f - p_f)^T\mathcal{B}_{ff}(\hat{p}_f - p_f) \\ &\quad - (\hat{p}_f - p_f)^T(\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a) \\ &\le -\lambda_{\min}\|\hat{p}_f - p_f\|^2 + \|\hat{p}_f - p_f\|\|\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a\| \\ &\le \|\hat{p}_f - p_f\|\left(-\lambda_{\min}\|\hat{p}_f - p_f\| + \|\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a\|\right).\end{aligned} \tag{8}$$

Note that $\|\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a\| = \|\Delta_{ff}\hat{p}_f - \Delta_{ff}p_f + \Delta_{ff}p_f + \Delta_{fa}p_a\| \le \|\Delta_{ff}\|\|\hat{p}_f - p_f\| + \|\Delta_{ff}p_f + \Delta_{fa}p_a\|$. Substituting it into (8) gives

$$\begin{aligned}\dot{V} \le \|\hat{p}_f - p_f\|&\left(-\lambda_{\min}(\mathcal{B}_{ff})\|\hat{p}_f - p_f\|\right. \\ &\left.+\|\Delta_{ff}\|\|\hat{p}_f - p_f\| + \|\Delta_{ff}p_f + \Delta_{fa}p_a\|\right).\end{aligned} \tag{9}$$

Since $\hat{R}_i$ converges to R_i exponentially fast by Theorem 1, we know that $\hat{g}_{ij} \to g_{ij}$ and consequently $\Delta_{ff} \to 0$ and $\Delta_{fa} \to 0$ exponentially fast as $t \to \infty$. As a result, given any $\mu \in (0, \lambda_{\min})$ and $\gamma > 0$, there exists a finite time $T > 0$ such that, for all $t > T$, $\|\Delta_{ff}\| < \mu$ and $\|\Delta_{ff}p_f + \Delta_{fa}p_a\| < \gamma$. Then, inequality (9) becomes

$$\begin{aligned}\dot{V} &\le \|\hat{p}_f - p_f\|\left((\mu - \lambda_{\min})\|\hat{p}_f - p_f\| + \gamma\right) \\ &= -(\lambda_{\min} - \mu)\|\hat{p}_f - p_f\|\left(\|\hat{p}_f - p_f\| - \frac{\gamma}{\lambda_{\min} - \mu}\right).\end{aligned} \tag{10}$$

Denote $b := \gamma/(\lambda_{\min} - \mu) > 0$. It follows from (10) that $\dot{V} < 0$ if $\|\hat{p}_f - p_f\| > b$. As a result, if $\|\hat{p}_f(T) - p_f\| \le b$, then $\|\hat{p}_f(t) - p_f\| \le b$ for all $t \in [T, \infty)$; if $\|\hat{p}_f(T) - p_f\| > b$, then $\|\hat{p}_f(t) - p_f\|$ is nonincreasing and hence bounded for all $t \in [T, \infty)$. Finally, since T is finite and $\hat{p}_f$ would not diverge in finite time, $\hat{p}_f(t)$ is also bounded within $[0, T]$. Therefore, $\|\hat{p}_f - p_f\|$ is bounded for all t. The boundedness of $\|\hat{p}_f - p_f\|$ implies the boundedness of $\|\hat{p}_f\|$.

Part 3. Convergence: Note that

$$\|\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a\| \le \|\Delta_{ff}\|\|\hat{p}_f\| + \|\Delta_{fa}\|\|p_a\|.$$

Since $\hat{g}_{ij}$ converges to g_{ij}, we know $\|\Delta_{ff}\|$ and $\|\Delta_{fa}\|$ converge to zero. It follows from the boundedness of $\|\hat{p}_f\|$ that $\|\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a\| \to 0$ as $t \to \infty$. When $\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a = 0$, system (7) becomes

$$\dot{\hat{p}}_f = -\mathcal{B}_{ff}(\hat{p}_f - p_f). \tag{11}$$

It is obvious that $\hat{p}_f$ in (11) converges to p_f. It follows from (11) that the error dynamics of $\hat{p}_f - p_f$ are linear and hence input-to-state stable [27, pp. 175]. Then, $\hat{p}_f$ in (7) converges to p_f as $\Delta_{ff}\hat{p}_f + \Delta_{fa}p_a \to 0$. ■

IV. Discrete-Time Localization Algorithms

This section studies discrete-time algorithms for bearing-based network localization.

A. Discrete-Time Localization With Global Bearings

We first propose an algorithm to localize a network using bearings measured in a global reference frame.

Let $\hat{p}_i(k)$ be the estimate of p_i where $k \in \{1, 2, \dots\}$. For $i \in \mathcal{V}_a$, we have $\hat{p}_i(k) = p_i$ for all k. For follower $i \in \mathcal{V}_f$, the proposed localization algorithm is

$$\hat{p}_i(k+1) = \hat{p}_i(k) - W_i^{-1} \sum_{j\in\mathcal{N}_i} P_{g_{ij}} (\hat{p}_i(k) - \hat{p}_j(k)) \tag{12}$$

where

$$W_i = \sum_{j\in\mathcal{N}_i} P_{g_{ij}} + \Lambda_i \tag{13}$$

and $\Lambda_i \in \mathbb{R}^{d\times d}$ can be selected as any positive-definite matrix. The nonsingularity of W_i is always guaranteed because both $\sum_{j\in\mathcal{N}_i} P_{g_{ij}}$ and Λ_i are positive definite. The matrix $\sum_{j\in\mathcal{N}_i} P_{g_{ij}}$ being positive definite is because $\mathcal{B}_{ff}$ is positive definite and so is every diagonal block matrix of $\mathcal{B}_{ff}$.

As will be shown later, any positive definite Λ_i would guarantee estimation convergence. However, different Λ_i may lead to different convergence rate of the algorithm. As will be demonstrated by simulation later, larger Λ_i would lead to slower convergence rate because the incremental term $W_i^{-1}\sum_{j\in\mathcal{N}_i} P_{g_{ij}}[\hat{p}_i(k) - \hat{p}_j(k)]$ would be smaller.

As a special yet important case, select Λ_i to be

$$\Lambda_i = |\mathcal{N}_i| I_d - \sum_{j\in\mathcal{N}_i} P_{g_{ij}} + \alpha_i I_d \tag{14}$$

where α_i can be selected as any positive constant and $|\mathcal{N}_i|$ denotes the number of neighbors of node i. The matrix Λ_i in (14) is positive definite because $\Lambda_i = \sum_{j\in\mathcal{N}_i}(I_d - P_{g_{ij}}) + \alpha_i I_d = \sum_{j\in\mathcal{N}_i} g_{ij}g_{ij}^T + \alpha_i I_d > 0$. Substituting (14) into (13) gives $W_i = (|\mathcal{N}_i| + \alpha_i)I$. As a result, the localization algorithm in (12) becomes

$$\hat{p}_i(k+1) = \hat{p}_i(k) - \frac{1}{|\mathcal{N}_i| + \alpha_i} \sum_{j\in\mathcal{N}_i} P_{g_{ij}} (\hat{p}_i(k) - \hat{p}_j(k)). \tag{15}$$

Algorithm (15) will be useful to design localization algorithms based on locally measured bearings.

1) Convergence Analysis: Let $W = \text{blkdiag}(W_i) \in \mathbb{R}^{dn_f\times dn_f}$. The matrix-vector form of (12) is

$$\begin{aligned}\hat{p}_f(k+1) &= \hat{p}_f(k) - W^{-1}(\mathcal{B}_{ff}\hat{p}_f(k) + \mathcal{B}_{fa}p_a) \\ &= \left(I - W^{-1}\mathcal{B}_{ff}\right)\hat{p}_f(k) - W^{-1}\mathcal{B}_{fa}p_a.\end{aligned} \tag{16}$$

Denote the localization error as

$$\delta_{p_f}(k) := \hat{p}_f(k) - p_f = \hat{p}_f(k) + \mathcal{B}_{ff}^{-1}\mathcal{B}_{fa}p_a. \tag{17}$$

It then follows from (16) that:

$$\delta_{p_f}(k+1) = \left(I - W^{-1}\mathcal{B}_{ff}\right)\delta_{p_f}(k). \tag{18}$$

The discrete-time system in (18) is convergent if and only if $I - W^{-1}\mathcal{B}_{ff}$ is Schur, which means $\rho(I - W^{-1}\mathcal{B}_{ff}) < 1$, where $\rho(\cdot)$ denotes the spectrum radius of a matrix.

Split $\mathcal{B}_{ff}$ such that $\mathcal{B}_{ff} = D_{ff} - E_{ff}$ where $D_{ff} \in \mathbb{R}^{dn_f\times dn_f}$ and $E_{ff} \in \mathbb{R}^{dn_f\times dn_f}$ contain the diagonal and off-diagonal block matrices of $\mathcal{B}_{ff}$, respectively. In particular, the ith diagonal $d\times d$ block of D_{ff} is $[D_{ff}]_{ii} = \sum_{j\in\mathcal{N}_i} P_{g_{ij}}$ and the other entries of D_{ff} are zero. The ijth $d\times d$ block of E_{ff} is $[E_{ff}]_{ij} = P_{g_{ij}}$ if $j \in \mathcal{N}_i$ and the other entries of E_{ff} are zero.

We next present a useful lemma and then the convergence result of (18).

Lemma 2: The matrix $2D_{ff} - \mathcal{B}_{ff}$ is positive semidefinite.

Proof: Note that $2D_{ff} - \mathcal{B}_{ff} = 2D_{ff} - (D_{ff} - E_{ff}) = D_{ff} + E_{ff}$. Split D_{ff} as $D_{ff} = D_{ff}^a + D_{ff}^f$ where D_{ff}^a and D_{ff}^f are block diagonal matrices. The ith diagonal blocks of them are

$$\left[D_{ff}^a\right]_{ii} = \sum_{j\in\mathcal{N}_i\cap\mathcal{V}_a} P_{g_{ij}}, \quad \left[D_{ff}^f\right]_{ii} = \sum_{j\in\mathcal{N}_i\cap\mathcal{V}_f} P_{g_{ij}}.$$

For any vector $v = [\dots, v_i^T, \dots]^T \in \mathbb{R}^{dn_f}$, we have

$$\begin{aligned}v^T(2D_{ff} - \mathcal{B}_{ff})v &= v^T(D_{ff} + E_{ff})v \\ &= v^T D_{ff}^a v + v^T(D_{ff}^f + E_{ff})v.\end{aligned}$$

It can be verified that

$$v^T D_{ff}^a v = \sum_{i\in\mathcal{V}_f} v_i^T \left(\sum_{j\in\mathcal{N}_i\cap\mathcal{V}_a} P_{g_{ij}}\right) v_i,$$

$$v^T(D_{ff}^f + E_{ff})v = \sum_{i\in\mathcal{V}_f}\sum_{j\in\mathcal{N}_i\cap\mathcal{V}_f} (v_i + v_j)^T P_{g_{ij}} (v_i + v_j).$$

As a result, $v^T D_{ff}^a v \geq 0$ and $v^T(D_{ff}^f + E_{ff})v \geq 0$, and consequently $v^T(2D_{ff} - \mathcal{B}_{ff})v \geq 0$. Therefore, $2D_{ff} - \mathcal{B}_{ff}$ is positive semidefinite. ■

Theorem 3 (Network Localization Convergence): If $(\mathcal{G}, p)$ is bearing localizable, given arbitrary initial estimates $\{\hat{p}_i(0)\}_{i\in\mathcal{V}_f}$, $\hat{p}_i(k)$ converges to p_i globally and exponentially fast for all $i \in \mathcal{V}_f$ under the action of (12) if $\Lambda_i > 0$ for all $i \in \mathcal{V}_f$.

Proof: Let $\Lambda = \text{blkdiag}(\Lambda_i) \in \mathbb{R}^{dn_f\times dn_f}$. Then, $W = D_{ff} + \Lambda$. It follows from Lemma 2 that $\mathcal{B}_{ff} \leq 2D_{ff}$, which further implies that $\mathcal{B}_{ff} < 2D_{ff} + 2\Lambda = 2W$ when $\Lambda > 0$. Multiplying $W^{-\frac{1}{2}}$ on both sides of the inequality of $\mathcal{B}_{ff} < 2W$ leads to

$$W^{-\frac{1}{2}}\mathcal{B}_{ff}W^{-\frac{1}{2}} < 2I.$$

Since $W^{-\frac{1}{2}}\mathcal{B}_{ff}W^{-\frac{1}{2}}$ is positive definite, its eigenvalues are real and positive. The above inequality suggests that the eigenvalues of $W^{-\frac{1}{2}}\mathcal{B}_{ff}W^{-\frac{1}{2}}$ are located in the interval of (0,2). Note that $W^{-\frac{1}{2}}\mathcal{B}_{ff}W^{-\frac{1}{2}}$ and $W^{-1}\mathcal{B}_{ff}$ have the same spectrum because they can be obtained from each other by similar transformations. In particular, $W^{-\frac{1}{2}}\mathcal{B}_{ff}W^{-\frac{1}{2}} = W^{\frac{1}{2}}(W^{-1}\mathcal{B}_{ff})W^{-\frac{1}{2}}$. As a result, the eigenvalues of $W^{-1}\mathcal{B}_{ff}$ are also real and located in (0,2). Hence, the eigenvalues of $I - W^{-1}\mathcal{B}_{ff}$ are real and located in $(-1, 1)$. Therefore, $\rho(I - W^{-1}\mathcal{B}_{ff}) < 1$ and system (18) is convergent. Since (18) is a linear system, it converges exponentially fast [28, Th. 5.9]. ■

In Theorem 3, the condition that $\Lambda_i > 0$ is merely sufficient but not necessary for the system convergence. When $\Lambda_i = 0$ for all i, the algorithm is still convergent when the network satisfies $2D_{ff} - \mathcal{B}_{ff} > 0$, as can be seen from the proof of Theorem 3.

It follows from Theorem 3 that algorithm (15) is convergent when $\alpha_i > 0$, because $\Lambda_i > 0$ when $\alpha_i > 0$.

B. Discrete-Time Localization With Local Bearings

This subsection considers the discrete-time case where all bearings are measured in local reference frames.

First of all, we present a discrete-time orientation estimation algorithm. For $i \in \mathcal{V}_a$, let $\hat{R}_i(k) = R_i$ for all k. For $i \in \mathcal{V}_f$, the estimation algorithm is

$$\hat{R}_i(k+1) = \hat{R}_i(k) - \frac{1}{w_i}\sum_{j\in\mathcal{N}_i} a_{ij}\left[\hat{R}_i(k) - R_iR_j^T\hat{R}_j(k)\right] \tag{19}$$

where

$$w_i = \sum_{j\in\mathcal{N}_i} a_{ij} + \beta_i.$$

In the above equations, a_{ij} is a positive constant weight for edge $(i, j) \in \mathcal{E}$ and $a_{ij} = a_{ji}$, and β_i can be selected as any positive constant. Note that $\hat{R}_i(k)$ in (19) may not be a rotation matrix.

The convergence of (19) is analyzed below. Let $L \in \mathbb{R}^{n\times n}$ be the graph Laplacian with a_{ij} as the weight for edge $(i, j) \in \mathcal{E}$. Partition L according to the partition of anchors and followers as

$$L = \begin{bmatrix} L_{aa} & L_{af} \\ L_{fa} & L_{ff} \end{bmatrix}$$

where $L_{fa} \in \mathbb{R}^{n_f\times n_a}$ and $L_{ff} \in \mathbb{R}^{n_f\times n_f}$.

Theorem 4 (Orientation Estimation Convergence): If $(\mathcal{G}, p)$ is connected and undirected, given arbitrary initial estimates $\{\hat{R}_i(0)\}_{i\in\mathcal{V}_f}$, $\hat{R}_i(k)$ converges to R_i globally and exponentially fast for all $i \in \mathcal{V}_f$ under the action of (19) if $\beta_i > 0$ for all $i \in \mathcal{V}_f$.

Proof: Multiplying R_i^T on both sides of (19) gives

$$R_i^T\hat{R}_i(k+1) = R_i^T\hat{R}_i(k) - \frac{1}{w_i}\sum_{j\in\mathcal{N}_i} a_{ij}\left(R_i^T\hat{R}_i(k) - R_j^T\hat{R}_j(k)\right). \tag{20}$$

Let $X_i(k) = R_i^T\hat{R}_i(k)$ and $x_{i,\ell}(k) \in \mathbb{R}^d$ be the ℓth column of $X_i(k)$ where $\ell = 1, \ldots, d$. Then, (20) can be rewritten as

$$x_{i,\ell}(k+1) = x_{i,\ell}(k) - \frac{1}{w_i}\sum_{j\in\mathcal{N}_i} a_{ij}(x_{i,\ell}(k) - x_{j,\ell}(k)). \tag{21}$$

Equation (21) is similar to the discrete-time containment control algorithms (i.e., consensus algorithms with multiple anchors) [29]. The difference lies in w_i. Since the interactions among the followers are undirected and connected, the entire graph has a united spanning tree. As a result, L_{ff} is nonsingular and $-L_{ff}^{-1}L_{fa}\mathbf{1}_{n_f} = \mathbf{1}_{n_a}$ [26, Lemma 5.1]. Then, (21) can be written in a matrix-vector form as

$$x_f(k+1) = x_f(k) - W^{-1}(L_{ff}\otimes I_d)x_f(k) - W^{-1}(L_{fa}\otimes I_d)x_a$$

where $W = \text{blkdiag}(w_iI_d) \in \mathbb{R}^{dn_f\times dn_f}$, $x_a = [x_1^T, \ldots, x_{n_a}^T]^T \in \mathbb{R}^{dn_a}$, and $x_f = [x_{n_a+1}^T, \ldots, x_n^T]^T \in \mathbb{R}^{dn_f}$. The subscript ℓ is dropped in the above equation for the sake of simplicity. Now define the error state as $\delta_f(k) = x_f(k) + (L_{ff}^{-1}L_{fa}\otimes I_d)x_a$. It then can be obtained that $\delta_f(k+1) = [I_{dn_f} - W^{-1}(L_{ff}\otimes I_d)]\delta_f(k)$. Similar to the proof of Theorem 3, it can be shown that $\rho(I_{dn_f} - W^{-1}(L_{ff}\otimes I_d)) < 1$. As a result, $\delta_f(k)$ converges to zero. Since it is linear system, it converges exponentially fast [28, Th. 5.9]. ■

Now we propose a discrete-time localization algorithm based on (15) and (19). For anchor $i \in \mathcal{V}_a$, $\hat{p}_i(k) = p_i$. For follower $i \in \mathcal{V}_f$, the proposed localization algorithm is

$$\hat{p}_i(k+1) = \hat{p}_i(k) - \frac{1}{|\mathcal{N}_i| + \alpha_i}\sum_{j\in\mathcal{N}_i} P_{\hat{g}_{ij}(k)}(\hat{p}_i(k) - \hat{p}_j(k)) \tag{22}$$

where

$$P_{\hat{g}_{ij}(k)} = I_d - \hat{g}_{ij}(k)\hat{g}_{ij}^T(k), \quad \hat{g}_{ij}(k) = \hat{R}_i(k)g_{ij}^{(i)}$$

and $\hat{R}_i(k)$ is governed by (19). It should be noted that, since $\hat{R}_i(k)$ may not be a rotation matrix, $\hat{g}_{ij}(k)$ may not be a unit vector and $P_{\hat{g}_{ij}(k)}$ may not be positive definite.

The convergence of the proposed localization algorithm is analyzed below.

Theorem 5 (Network Localization Convergence): If $(\mathcal{G}, p)$ is bearing localizable, given arbitrary initial estimates $\{\hat{p}_i(0)\}_{i\in\mathcal{V}_f}$ and $\{\hat{R}_i(0)\}_{i\in\mathcal{V}_f}$, $\hat{p}_i(k)$ converges to p_i globally asymptotically for all $i \in \mathcal{V}_f$ under the action of (22) and (19) if $\alpha_i, \beta_i > 0$ for all $i \in \mathcal{V}_f$.

Proof: The proof consists of three parts.

Part 1. Matrix form: Rewrite (22) in a matrix-vector form as

$$\hat{p}_f(k+1) = \hat{p}_f(k) - W^{-1}(\hat{\mathcal{B}}_{ff}\hat{p}_f(k) + \hat{B}_{fa}p_a) \tag{23}$$

where $W = \text{blkdiag}((|\mathcal{N}_i| + \alpha_i)I_d) \in \mathbb{R}^{dn_f\times dn_f}$, and $\hat{\mathcal{B}}_{ff}$ and $\hat{\mathcal{B}}_{fa}$ are obtained by replacing g_{ij} by $\hat{g}_{ij}$ in $\mathcal{B}_{ff}$ and $\mathcal{B}_{fa}$, respectively. Define $\Delta_{ff} := \hat{\mathcal{B}}_{ff} - \mathcal{B}_{ff}$ and $\Delta_{fa} := \hat{\mathcal{B}}_{fa} - \mathcal{B}_{fa}$. Then, (23) can be expressed as

$$\hat{p}_f(k+1) = \hat{p}_f(k) - W^{-1}(\mathcal{B}_{ff}\hat{p}_f(k) + B_{fa}p_a) - W^{-1}(\Delta_{ff}\hat{p}_f(k) + \Delta_{fa}p_a)$$

Then, the error $\delta_f(k)$ as defined in (17) satisfies $\delta_f(k+1) = (I - W^{-1}B_{ff})\,\delta_f(k) - W^{-1}\,(\Delta_{ff}\hat{p}_f(k) + \Delta_{fa}p_a) = (I - W^{-1}\mathcal{B}_{ff})\delta_f(k) - W^{-1}(\Delta_{ff}\hat{p}_f(k) - \Delta_{ff}p_f + \Delta_{ff}p_f + \Delta_{fa}p_a) = (I - W^{-1}\mathcal{B}_{ff})\;\delta_f(k)\; - W^{-1}\;\Delta_{ff}\;\delta_f(k) - W^{-1}(\Delta_{ff}p_f +$

$\Delta_{fa} p_a)$. For the sake of simplicity, let $A := I - W^{-1}\mathcal{B}_{ff}$ and $h := -W^{-1}(\Delta_{ff} p_f + \Delta_{fa} p_a)$. Then, the error dynamics can be written as

$$\delta_f(k+1) = A\delta_f(k) - W^{-1}\Delta_{ff}\delta_f(k) + h. \tag{24}$$

Part 2. Boundedness: Consider any positive definite matrix $Q \in \mathbb{R}^{dn_f \times dn_f}$. Since A is Schur, there exists a unique positive-definite matrix $P \in \mathbb{R}^{dn_f \times dn_f}$ such that $A^T PA - P = -Q$ [28]. Consider the Lyapunov function

$$V(k) = \delta^T(k)P\delta(k).$$

It follows from (24) that:

$$\begin{aligned}
&V(k+1) - V(k)\\
&= \delta^T(k+1)P\delta(k+1) - \delta^T(k)P\delta(k)\\
&= (A\delta_f(k) - W^{-1}\Delta_{ff}\delta_f(k) + h)^T P\\
&\quad \times (A\delta_f(k) - W^{-1}\Delta_{ff}\delta_f(k) + h) - \delta^T(k)P\delta(k)\\
&= \delta_f^T(k)A^T PA\delta_f(k) - \delta^T(k)P\delta(k)\\
&\quad + (-W^{-1}\Delta_{ff}\delta_f(k) + h)^T P(-W^{-1}\Delta_{ff}\delta_f(k) + h)\\
&\quad + 2\delta_f^T(k)A^T P(-W^{-1}\Delta_{ff}\delta_f(k) + h)\\
&= -\delta_f^T(k)Q\delta_f(k)\\
&\quad + (W^{-1}\Delta_{ff}\delta_f(k) - h)^T P(W^{-1}\Delta_{ff}\delta_f(k) - h)\\
&\quad + 2\delta_f^T(k)A^T P(-W^{-1}\Delta_{ff}\delta_f(k) + h).
\end{aligned}$$

Note that $-\delta_f^T(k)Q\delta_f(k) \le -\lambda_{\min}(Q)\|\delta_f(k)\|^2$. Based on the inequalities of vector norms, the above equation implies

$$\begin{aligned}
V(k+1) - V(k) \le -\lambda_{\min}(Q)\|\delta_f(k)\|^2 + \eta_1(k)\|\delta_f(k)\|^2\\
+ \eta_2(k)\|\delta_f(k)\| + \eta_3(k)
\end{aligned}$$

where the right-hand side is a quadratic function of $\|\delta_f(k)\|$. The expressions of η_1, η_2, and η_3 can be easily obtained and omitted here. It must be noted that η_1, η_2, and η_3 are all functions of the norms of Δ_{ff} and Δ_{fa}. Since $\hat{R}_i(k)$ converges to R_i exponentially fast, we know that $\hat{g}_{ij} \to g_{ij}$, $\|\Delta_{ff}\|, \|\Delta_{fa}\| \to 0$, and consequently $\eta_1, \eta_2, \eta_3 \to 0$ exponentially fast. As a result, for any constants $c_1, c_2, c_3 > 0$ where $c_1 \le \lambda_{\min}(Q)$, there exists a positive integer m such that, for all $k \ge m$, we have $\eta_1(k) \le c_1$, $\eta_2(k) \le c_2$, and $\eta_3(k) \le c_3$. Then, we have

$$\begin{aligned}
&V(k+1) - V(k)\\
&\le -\lambda_{\min}(Q)\|\delta_f(k)\|^2 + c_1\|\delta_f(k)\|^2 + c_2\|\delta_f(k)\| + c_3,\\
&= (-\lambda_{\min}(Q) + c_1)\|\delta_f(k)\|^2 + c_2\|\delta_f(k)\| + c_3.
\end{aligned}$$

Since $-\lambda_{\min}(Q) + c_1 < 0$, the right-hand side of the above inequality is negative when $\|\delta_f(k)\| \ge (c_2 + \sqrt{c_2^2 + 2c_3(\lambda_{\min}(Q) - c_1)})\ /\ (2(\lambda_{\min}\ (Q) - c_1)) := r > 0$. As a result, when $\|\delta_f(k)\| \ge r$, the Lyapunov function $V(k)$ is nonincreasing and remains bounded, which implies that $\|\delta_f(k)\|$ is bounded for $k \ge m$. Since $\|\delta_f(k)\|$ is bounded for all $k < m$, it is bounded for all k.

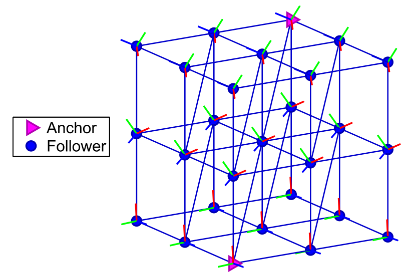

Fig. 1. True three-dimensional network to be localized. The red-green-blue lines represent the body frame for each node.

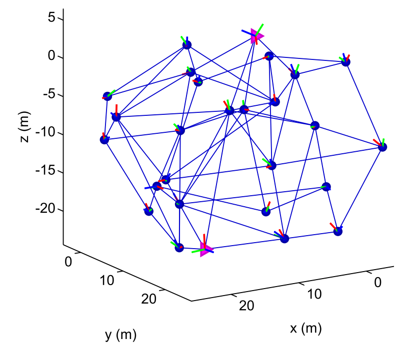

Fig. 2. Initial estimates of the positions and orientations of the nodes. Note that the initial estimates of the orientations are not required to be rotation matrices.

Part 3. Convergence: Since $\|\delta_f\|$ is bounded, the term $-W^{-1}\Delta_{ff}\delta_f(k) + h$ in (24) converges to zero. Since (24) without the term of $-W^{-1}\Delta_{ff}\delta_f(k) + h$ is linear time-invariant and hence input-to-state stable, we have $\delta_f(k) \to 0$ when $-W^{-1}\Delta_{ff}\delta_f(k) + h \to 0$ [30]. ■

V. Simulation Examples

Two simulation examples are given in this section to verify the effectiveness of the proposed algorithms.

The true network for the two simulation examples is shown in Fig. 1. This network consists of 27 nodes, where two of them are anchors and the rest 25 nodes are followers. The red-green-blue lines represent the body frames for the nodes. As shown in Fig. 1, the nodes located in the same horizontal layer have the same orientation, but nodes in different layers have different orientations. The random initial estimates of the positions and orientations of the followers are given in Fig. 2. Note that the initial estimates of the orientations could be arbitrary matrices rather than rotation matrices.

Simulation results for the continuous-time orientation estimation and localization algorithms in (2) and (5) are shown in Fig. 3. As can be seen in Fig. 3(a)–(b), both orientation and localization errors converge to zero eventually. Fig. 3(c) shows the simulation results when the Gram–Schmidt orthogonalization procedure in [14] is used. In particular, $\hat{R}_i$ is orthogonalized and then substituted to calculate $\hat{g}_{ij}$ in (5), whereas algorithm (2) is unchanged. As can be seen, the orthogonalization does not change the convergence rate significantly. However, it brings

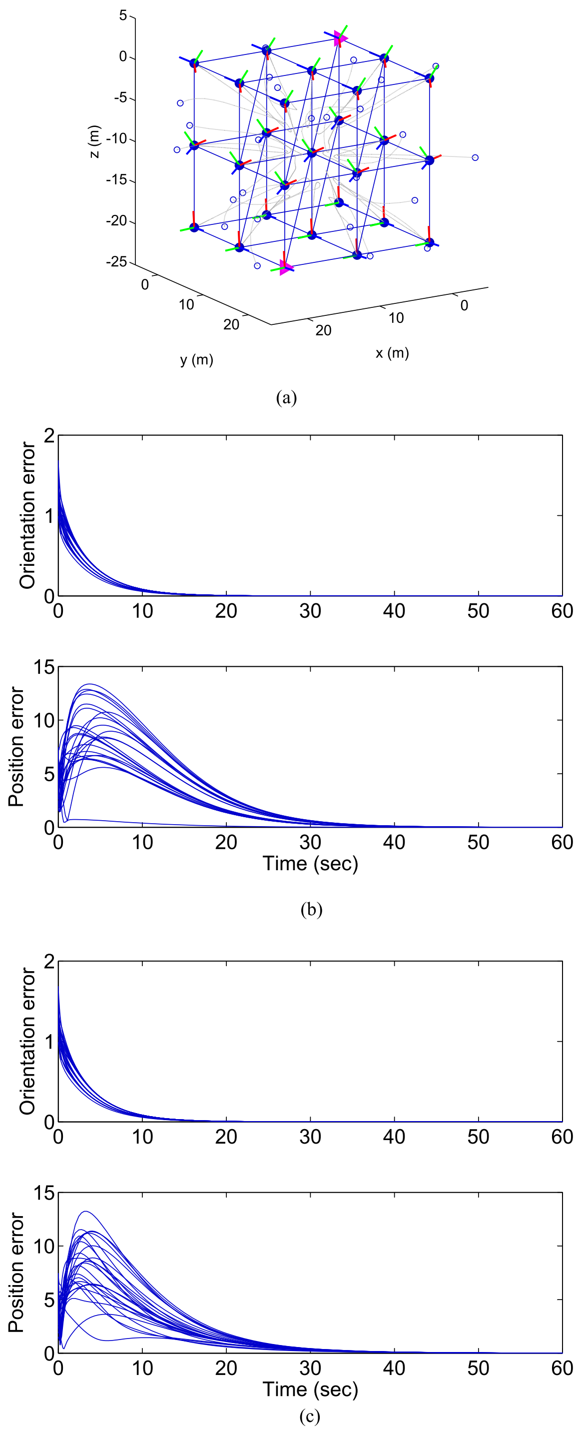

Fig. 3. Simulation results for the continuous-time algorithms in (5) and (2). (a) Initial estimates (hollow dots) and final estimates (solid dots). (b) Orientation error: $\|\hat{R}_i - R_i\|$; position error: $\|\hat{p}_i - p_i\|$. (c) The orthogonalization procedure in [14] is added to (5).

additional problems. For example, the global convergence property is invalid and the orthogonalization would fail when $\hat{R}_i$ is singular.

Simulation results for the discrete-time orientation estimation and localization algorithms in (19) and (22) are shown in Fig. 4.

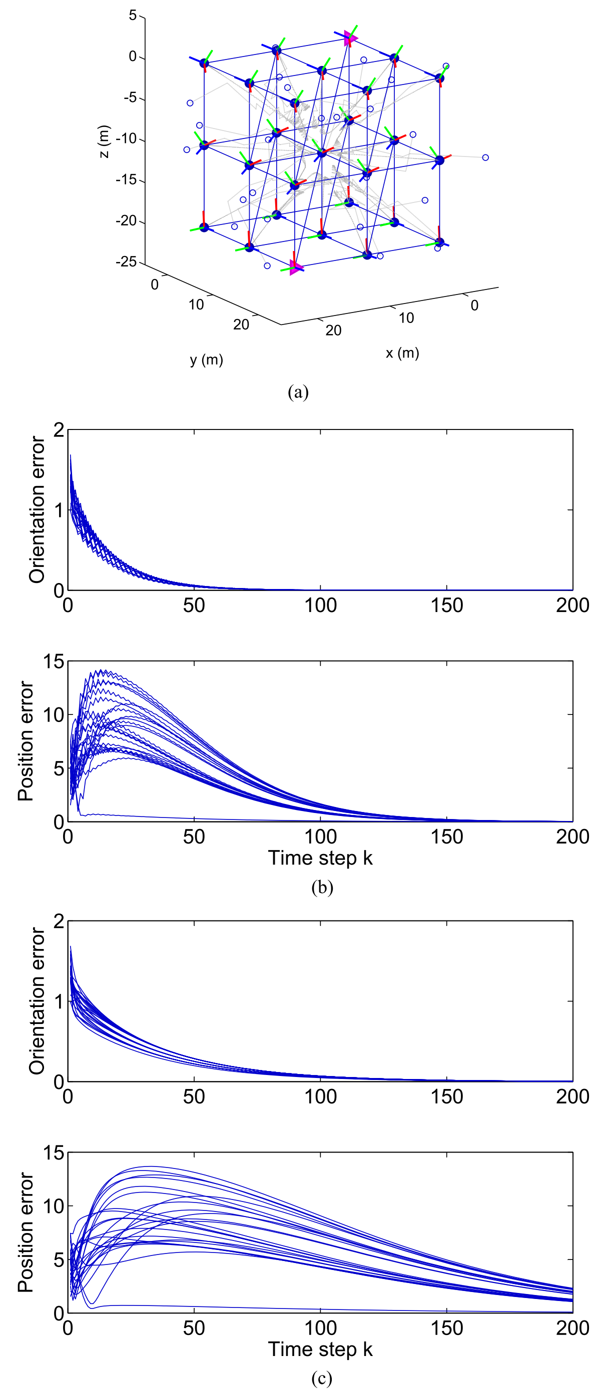

Fig. 4. Simulation results for the discrete-time algorithms in (22) and (19). Orientation error: $\|\hat{R}_i - R_i\|$; position error: $\|\hat{p}_i - p_i\|$. (a) Initial estimates (hollow dots) and final estimates (solid dots). (b) $\alpha_i = 0$ and $\beta_i = 0$ for all $i \in \mathcal{V}_f$. (c) $\alpha_i = 5$ and $\beta_i = 5$ for all $i \in \mathcal{V}_f$.

Fig. 4(a)–(b) show the results when α_i and β_i in (19) and (22) are selected to be zero for all $i \in \mathcal{V}_f$. As can be seen, both orientation and localization errors converge to zero eventually. It verifies that $\alpha_i > 0$ and $\beta_i > 0$ are sufficient but not necessary to ensure estimation convergence. Fig. 4(c) shows the simulation results when $\alpha_i = 5$ and $\beta_i = 5$. As can be seen, the system is still convergent, but the convergence rate is slower, which verifies that large α_i and β_i would slow down the convergence rate.

VI. Conclusion

In this paper, we presented both continuous-time and discrete-time distributed algorithms to solve the problem of network localization using locally measured bearings. The proposed algorithms consist of coupled orientation and position estimation procedures. It has been shown that when a network is bearing localizable, the orientation and position estimation errors converge to zero given arbitrary initial estimates. The results in this paper could be generalized in different directions by considering more realistic scenarios where the network is directed and switching and suffers from measurement noises or time delays. In particular, this paper only considered the case of accurate measurements. In practice, the measurements may be corrupted by measurement errors. There are two different sources of measurement errors. The first is the measurement errors in the anchors' absolute positions, and the second is the measurement errors of the relative orientations and anchors' absolute orientations. If there are no orientation errors, bounded anchor position errors would result in bounded localization estimation error since the estimation problem is a linear system. However, when there are orientation errors, the problem becomes much more complex and the estimation convergence may be jeopardized [10]. The tolerance of a network to orientation errors deserves further study in the future.

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