Laman graphs are generically bearing rigid in arbitrary dimensions

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What is bearing rigidity?

Revisit distance rigidity:
◊ If we fix the length of each edge in a network, can the geometric pattern of the network be uniquely determined?
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◇ If we fix the length of each edge in a network, can the geometric pattern of the network be uniquely determined?

Bearing rigidity:
◇ If we fix the bearing of each edge in a network, can the geometric pattern of the network be uniquely determined?

Loose definition: a network bearing rigid if its bearings can uniquely determine its geometric pattern.
Why study bearing rigidity?

- Initially: computer-aided graphical drawing [Servatius and Whiteley, 1999]

Network localization:

Formation control:
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◊ Initially: computer-aided graphical drawing [Servatius and Whiteley, 1999]
◊ Network localization:

◊ Formation control:
Two key problems in bearing rigidity theory

- How to determine the bearing rigidity of a given network?
- How to construct a bearing rigid network from scratch?
Notations for Bearing Rigidity

◊ Notations:

- **Graph**: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- **Configuration**: $p_i \in \mathbb{R}^d$ with $i \in \mathcal{V}$ and $p = [p_1^T, \ldots, p_n^T]^T$.
- **Network**: graph+configuration

An orthogonal projection matrix:

$$P_{ij} = I_d - g_{ij} g_{ij}^T$$
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◊ Bearing:

$$g_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|} \quad \forall (i, j) \in \mathcal{E}.$$  

Example:

![Diagram](image-url)
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◊ An orthogonal projection matrix:

\[ P_{g_{ij}} = I_d - g_{ij}g_{ij}^T. \]
Notations for Bearing Rigidity

⋄ Properties:

- $P_{g_{ij}}$ is symmetric positive semi-definite and $P_{g_{ij}}^2 = P_{g_{ij}}$
- $\text{Null}(P_{g_{ij}}) = \text{span}\{g_{ij}\} \iff P_{g_{ij}} x = 0 \text{ iff } x \parallel g_{ij}$ (important)
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◇ Bearing Laplacian: $\mathcal{B} \in \mathbb{R}^{dn \times dn}$ with the $ij$th subblock matrix as

$$[\mathcal{B}]_{ij} = \begin{cases} 0_{d \times d}, & i \neq j, (i, j) \notin \mathcal{E} \\ -P_{g_{ij}}, & i \neq j, (i, j) \in \mathcal{E} \\ \sum_{j \in \mathcal{N}_i} P_{g_{ij}}, & i \in \mathcal{V} \end{cases}$$
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\end{cases}$$

**Example:**

$$\mathcal{B} = \begin{bmatrix}
P_{g_{12}} + P_{g_{13}} & -P_{g_{12}} & -P_{g_{13}} \\
-P_{g_{21}} & P_{g_{21}} + P_{g_{23}} & -P_{g_{23}} \\
-P_{g_{31}} & -P_{g_{32}} & P_{g_{31}} + P_{g_{32}}
\end{bmatrix}$$
Examine the bearing rigidity of a given network

Condition for Bearing Rigidity [Zhao and Zelazo, 2016b]

A network is bearing rigid if and only if

\[ \text{rank}(\mathcal{B}) = dn - d - 1 \]

Proof.

\[ f(p) \triangleq \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix} \in \mathbb{R}^{dm}. \]

\[ R(p) \triangleq \frac{\partial f(p)}{\partial p} \in \mathbb{R}^{dm \times dn}. \]

\[ df(p) = R(p)dp \]

Trivial motions: translation and scaling
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Trivial motions: translation and scaling

○ Examples of bearing rigid networks:

○ Examples of networks that are not bearing rigid:
Construction of bearing rigid networks

- Importance: construct sensor networks and formation
- Need to design graph $G$ and configuration $p$
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  ![Graph examples]

- Intuitively, it seems configuration is not that important. Is it true?
Definition (**Generically Bearing Rigid Graphs**)

A graph $\mathcal{G}$ is generically bearing rigid in $\mathbb{R}^d$ if there exists at least one configuration $p$ in $\mathbb{R}^d$ such that $(\mathcal{G}, p)$ is bearing rigid.
Construction of bearing rigid networks

Definition (Generically Bearing Rigid Graphs)

A graph $G$ is generically bearing rigid in $\mathbb{R}^d$ if there exists at least one configuration $p$ in $\mathbb{R}^d$ such that $(G, p)$ is bearing rigid.

Lemma (Density of Generical Bearing Rigid Graphs)

If $G$ is generically bearing rigid in $\mathbb{R}^d$, then $(G, p)$ is bearing rigid for almost all $p$ in $\mathbb{R}^d$ in the sense that the set of $p$ where $(G, p)$ is not bearing rigid is of measure zero. Moreover, for any configuration $p_0$ and any small constant $\epsilon > 0$, there always exists a configuration $p$ such that $(G, p)$ is bearing rigid and $\|p - p_0\| < \epsilon$. 
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**Summary:**

- If a graph is generically bearing rigid, then for any almost all configurations the corresponding network is bearing rigid.
- If a graph is not generically bearing rigid, by definition for any configuration the corresponding network is not bearing rigid.
Construction of bearing rigid graphs

- Construction of bearing rigid networks $\Rightarrow$ construction of bearing rigid graphs

Definition (Laman Graphs)

A graph $G = (V, E)$ is Laman if $|E| = 2|V| - 3$ and every subset of $k \geq 2$ vertices spans at most $2k - 3$ edges.

Why consider Laman graphs:
- Favorable since edges distribute evenly in a Laman graph;
- Widely used in, for example, distance rigidity;
- Can be constructed by Henneberg Construction.

Definition (Henneberg Construction)

Given a graph $G = (V, E)$, a new graph $G' = (V', E')$ is formed by adding a new vertex $v$ to $G$ and performing one of the following two operations:

(a) Vertex addition: connect vertex $v$ to any two existing vertices $i, j \in V$. In this case, $V' = V \cup \{v\}$ and $E' = E \cup \{(v, i), (v, j)\}$.

(b) Edge splitting: consider three vertices $i, j, k \in V$ with $(i, j) \in E$ and connect vertex $v$ to $i, j, k$ and delete $(i, j)$. In this case, $V' = V \cup \{v\}$ and $E' = E \cup \{(v, i), (v, j), (v, k)\} \setminus \{(i, j)\}$.
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- Construction of bearing rigid networks $\implies$ construction of bearing rigid graphs
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Construction of bearing rigid graphs

Two operations in Henneberg construction:

(a) Vertex addition

(b) Edge splitting

Figure: The procedure to construct a three-dimensional bearing rigid network. The number of edges in this network is equal to $2n - 3 = 13$. 
Construction of bearing rigid graphs

Two operations in Henneberg construction:

Example:

Step 1: vertex addition

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(b) Edge splitting

Example:

Step 1: vertex addition

Step 2: edge splitting

\[
E = 2n - 3 = 13.
\]
Construction of bearing rigid graphs

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Example:

Step 1: vertex addition

Step 2: edge splitting

Step 3: edge splitting

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Construction of bearing rigid graphs

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Example:

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Step 5: edge splitting
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Example:

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Step 5: edge splitting

Step 6: edge splitting
Construction of bearing rigid graphs

Theorem (Main Result)

Laman graphs are generically bearing rigid in arbitrary dimensions.

◊ Rephrase the main result: If a graph is Laman, then for almost all configurations the corresponding network is bearing rigid.
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Laman graphs are generically bearing rigid in arbitrary dimensions.

◊ Rephrase the main result: If a graph is Laman, then for almost all configurations the corresponding network is bearing rigid.

**Proof.**

Partition $\mathcal{B}$ into

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{bmatrix},$$

where $\mathcal{B}_{22} \in \mathbb{R}^{2d \times 2d}$ corresponds to nodes $i, j$. Then $\mathcal{B}'$ can be expressed as

$$\mathcal{B}' = \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} + D & F \\ 0 & -F^T & -E \end{bmatrix},$$

where
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Yes, in $\mathbb{R}^2$

Theorem

A graph is bearing rigid in $\mathbb{R}^2$ if and only if the graph contains a Laman spanning subgraph.
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**Theorem**

A graph is bearing rigid in $\mathbb{R}^2$ if and only if the graph contains a Laman spanning subgraph.

No, in higher dimensions
Two key problems in the bearing rigidity theory:

- How to examine the bearing rigidity of a given network?
  - Bearing Laplacian
  - Rank condition
- How to construct a bearing rigid network?
  - Graph is critical
  - Laman graphs are generically bearing rigid


