Laman graphs are generically bearing rigid in arbitrary dimensions

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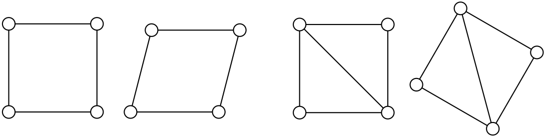
December 2017

What is bearing rigidity?

Revisit distance rigidity:
⋄ If we fix the length of each edge in a network, can the geometric pattern of the network be uniquely determined?



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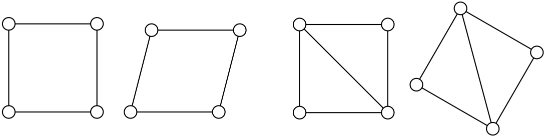
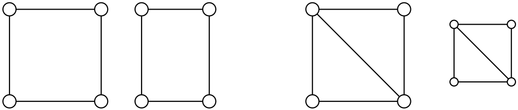


Bearing rigidity:

⋄ If we fix the bearing of each edge in a network, can the geometric pattern of the network be uniquely determined?



Loose definition: a network bearing rigid if its bearings can uniquely determine its geometric pattern.

Why study bearing rigidity?

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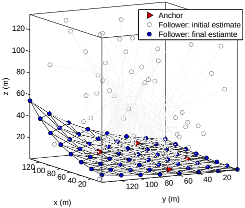
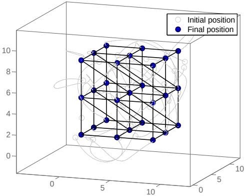
◇ Network localization:



◇ Formation control:



Two key problems in bearing rigidity theory

- How to determine the bearing rigidity of a given network?
- How to construct a bearing rigid network from scratch?

Notations for Bearing Rigidity

◇ Notations:

- Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$
- Configuration: $p_i \in \mathbb{R}^d$ with $i \in \mathcal{V}$ and $p = [p_1^{\mathrm{T}}, \ldots, p_n^{\mathrm{T}}]^{\mathrm{T}}$.
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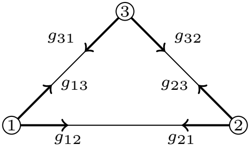
◇ Bearing:

$$g_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|} \quad \forall (i,j) \in \mathcal{E}.$$

Example:



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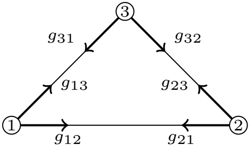
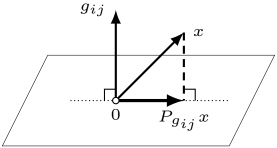


◇ An orthogonal projection matrix:

$$P_{g_{ij}} = I_d - g_{ij} g_{ij}^T,$$

Notations for Bearing Rigidity

◇ Properties:



- $P_{g_{ij}}$ is symmetric positive semi-definite and $P_{g_{ij}}^2 = P_{g_{ij}}$
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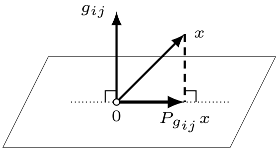

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◇ Bearing Laplacian: $\mathcal{B} \in \mathbb{R}^{dn \times dn}$ with the ijth subblock matrix as

$$[\mathcal{B}]_{ij} = \begin{cases} \mathbf{0}_{d\times d}, & i \neq j, (i,j) \notin \mathcal{E} \\ -P_{g_{ij}}, & i \neq j, (i,j) \in \mathcal{E} \\ \sum_{j\in\mathcal{N}_i} P_{g_{ij}}, & i \in \mathcal{V} \end{cases}$$

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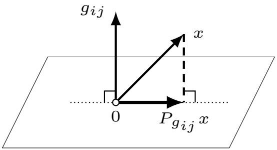

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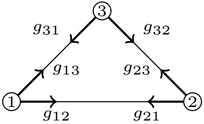

$$\mathcal{B} = \begin{bmatrix} P_{g_{12}} + P_{g_{13}} & -P_{g_{12}} & -P_{g_{13}} \\ -P_{g_{21}} & P_{g_{21}} + P_{g_{23}} & -P_{g_{23}} \\ -P_{g_{31}} & -P_{g_{32}} & P_{g_{31}} + P_{g_{32}} \end{bmatrix}$$

Examine the bearing rigidity of a given network

Condition for Bearing Rigidity [Zhao and Zelazo, 2016b]

A network is bearing rigid if and only if $\mathrm{rank}(\mathcal{B}) = dn - d - 1$

Proof.

$$f(p) \triangleq \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix} \in \mathbb{R}^{dm}.$$

$$R(p) \triangleq \frac{\partial f(p)}{\partial p} \in \mathbb{R}^{dm \times dn}.$$

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Trivial motions: translation and scaling □

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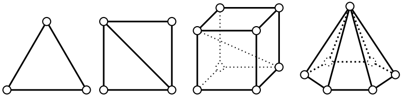
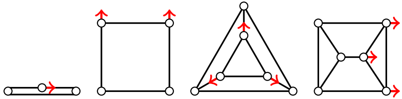
◇ Examples of bearing rigid networks:



(e) (f) (g) (h)

◇ Examples of networks that are not bearing rigid:



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Construction of bearing rigid networks

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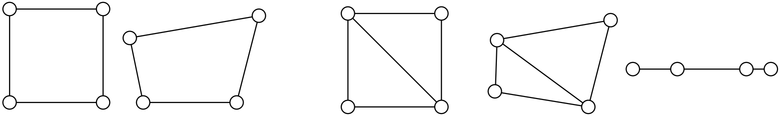

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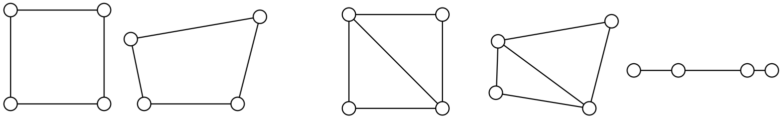

◇ Intuitively, it seems configuration is not that important. Is it true?

Construction of bearing rigid networks

Definition (Generically Bearing Rigid Graphs)

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Lemma (Density of Generical Bearing Rigid Graphs)

If $\mathcal{G}$ is generically bearing rigid in $\mathbb{R}^d$, then $(\mathcal{G}, p)$ is bearing rigid for almost all p in $\mathbb{R}^d$ in the sense that the set of p where $(\mathcal{G}, p)$ is not bearing rigid is of measure zero. Moreover, for any configuration p_0 and any small constant $\epsilon > 0$, there always exists a configuration p such that $(\mathcal{G}, p)$ is bearing rigid and $\|p - p_0\| < \epsilon$.

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Summary:

- If a graph is generically bearing rigid, then for any almost all configurations the corresponding network is bearing rigid.
- If a graph is not generically bearing rigid, by definition for any configuration the corresponding network is not bearing rigid.

Construction of bearing rigid graphs

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A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is Laman if $|\mathcal{E}| = 2|\mathcal{V}| - 3$ and every subset of $k \geq 2$ vertices spans at most $2k - 3$ edges.

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Definition (Henneberg Construction)

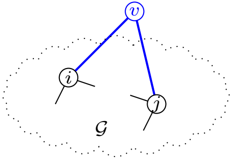
Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a new graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is formed by adding a new vertex v to $\mathcal{G}$ and performing one of the following two operations:

(a) *Vertex addition*: connect vertex v to any two existing vertices $i, j \in \mathcal{V}$. In this case, $\mathcal{V}' = \mathcal{V} \cup \{v\}$ and $\mathcal{E}' = \mathcal{E} \cup \{(v, i), (v, j)\}$.

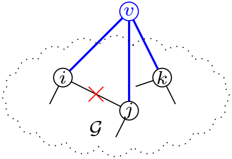
(b) *Edge splitting*: consider three vertices $i, j, k \in \mathcal{V}$ with $(i, j) \in \mathcal{E}$ and connect vertex v to i, j, k and delete (i, j). In this case, $\mathcal{V}' = \mathcal{V} \cup \{v\}$ and $\mathcal{E}' = \mathcal{E} \cup \{(v, i), (v, j), (v, k)\} \setminus \{(i, j)\}$.

Construction of bearing rigid graphs

Two operations in Henneberg construction:



(a) Vertex addition



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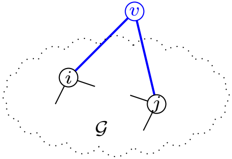

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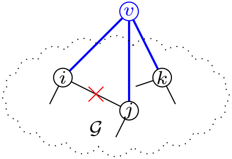

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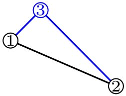
Example:



Step 1: vertex addition

Construction of bearing rigid graphs

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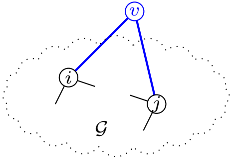

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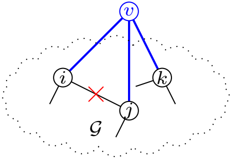

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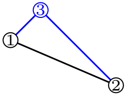

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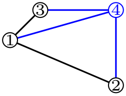

Step 2: edge splitting

Construction of bearing rigid graphs

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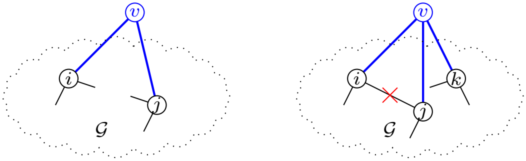

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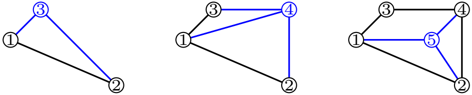

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Step 3: edge splitting

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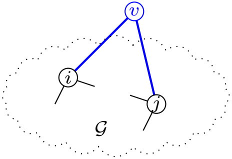

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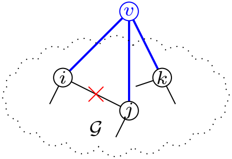

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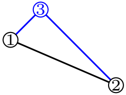

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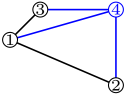

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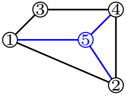

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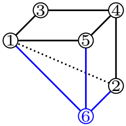
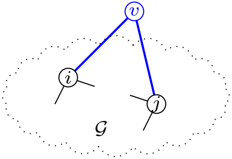


Step 4: edge splitting

Construction of bearing rigid graphs

Two operations in Henneberg construction:



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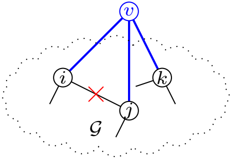

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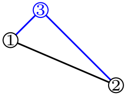

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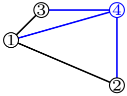

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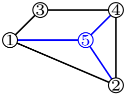

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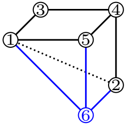

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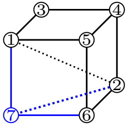

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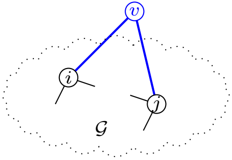

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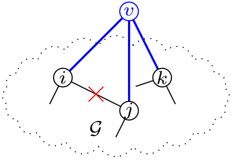

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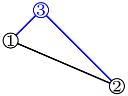
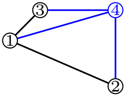
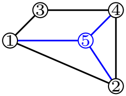


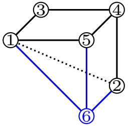
Step 1: vertex addition



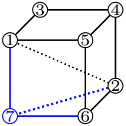
Step 2: edge splitting



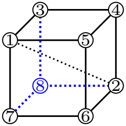
Step 3: edge splitting



Step 4: edge splitting



Step 5: edge splitting



Step 6: edge splitting

Construction of bearing rigid graphs

Theorem (Main Result)

Laman graphs are generically bearing rigid in arbitrary dimensions.

◇ Rephrase the main result: If a graph is Laman, then for almost all configurations the corresponding network is bearing rigid.

Construction of bearing rigid graphs

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$\diamond$ Rephrase the main result: If a graph is Laman, then for almost all configurations the corresponding network is bearing rigid.

Proof.

Partition $\mathcal{B}$ into

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \end{bmatrix},$$

where $\mathcal{B}_{22} \in \mathbb{R}^{2d \times 2d}$ corresponds to nodes i, j. Then $\mathcal{B}'$ can be expressed as

$$\mathcal{B}' = \left[\begin{array}{cc|c} \mathcal{B}_{11} & \mathcal{B}_{12} & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} + D & F \\ \hline 0 & F^T & E \end{array}\right],$$

□

Construction of bearing rigid graphs

◇ Question: is Laman both necessary and sufficient for bearing rigidity?

Construction of bearing rigid graphs

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- ◇ Yes, in $\mathbb{R}^2$

Theorem

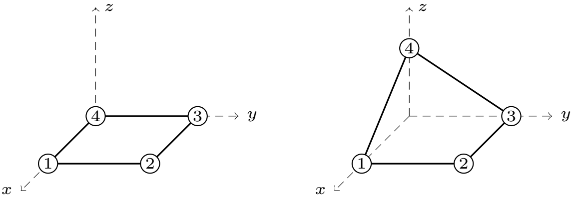
A graph is bearing rigid in $\mathbb{R}^2$ *if and only if the graph contains a Laman spanning subgraph.*

- ⋄ Question: is Laman both necessary and sufficient for bearing rigidity?
- ⋄ Yes, in $\mathbb{R}^2$

Theorem

A graph is bearing rigid in $\mathbb{R}^2$ if and only if the graph contains a Laman spanning subgraph.

- ⋄ No, in higher dimensions



(a) (b)

Conclusion

◇ Two key problems in the bearing rigidity theory:

- How to examine the bearing rigidity of a given network?
  - Bearing Laplacian
  - Rank condition
- How to construct a bearing rigid network?
  - Graph is critical
  - Laman graphs are generically bearing rigid

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