Distributed Affine Formation Maneuver Control of Multi-Agent Systems

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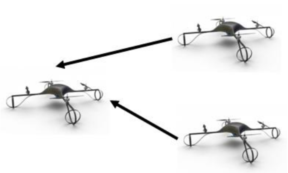
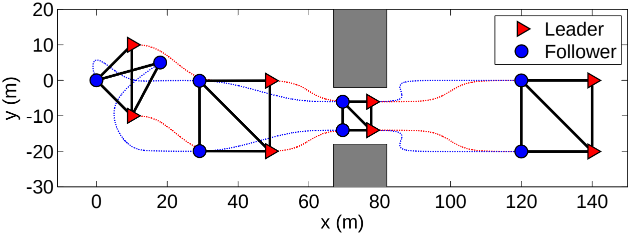
Introduction: What is formation control?



Formation control has two objectives:

- Formation shape control
- Formation maneuver control



Introduction: Various approaches

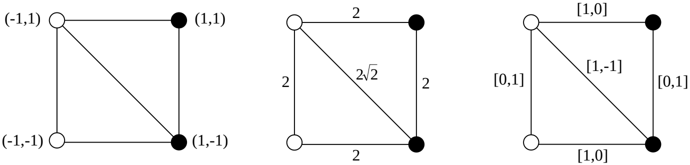
History of formation control:

- before consensus (2004)
- after consensus

Different ways to define target formations:

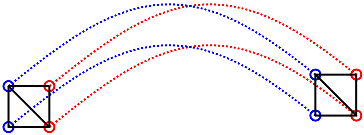
- inter-agent relative position
- inter-agent distance
- inter-agent bearing
- complex Laplacian
- barycentric coordinate
- stress matrix

Example:

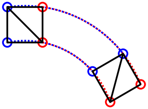


Introduction: Why so many approaches?

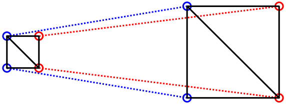
Different approaches lead to different maneuverability of the formation!



(a) Translational maneuver



(b) Translational and rotational maneuver



(c) Translational and scaling maneuver

Can we achieve all of them simultaneously?

Introduction: Our aim

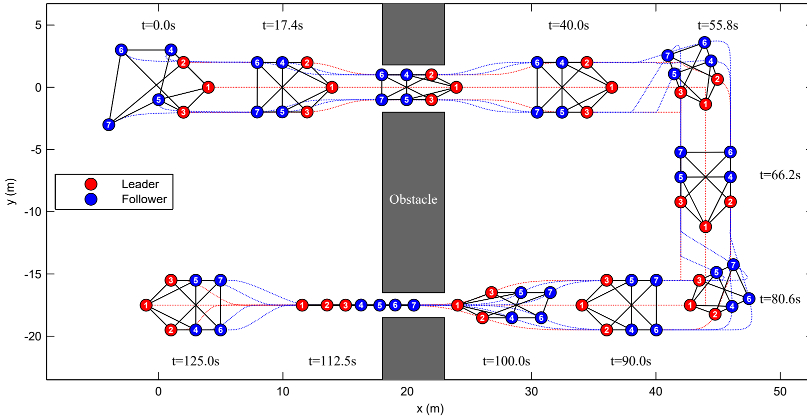


Figure: Our aim

Problem statement

◇ Notations:

- Positions of the agents: $p(t) = [p_1^T(t), \ldots, p_n^T(t)]^T$
- The first n_ℓ agents are *leaders* and the rest n_f agents are *followers*
- Underlying graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, undirected and fixed
- Each agent can obtain the relative positions of its neighbors

◇ Nominal formation: $p^d = [(p_1^d)^T, \ldots, (p_n^d)^T]^T$

◇ Time-varying target formation:

- Element-wise expression:

$$p_i^*(t) = A^d(t)p_i^d + b^d(t), \quad i = 1, \ldots, n$$

- Matrix-vector expression:

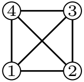
$$p^*(t) = [I_n \otimes A^d(t)]p^d + \mathbf{1}_n \otimes b^d(t)$$

Problem statement

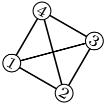
- ◇ Target formation:

$$p_i^*(t) = A^d(t)p_i^d + b^d(t), \quad i = 1, \ldots, n$$

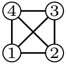
- ◇ Different cases:
  - translation: $A^d = I$ and $b^d(t)$ is arbitrary
  - rotation: A^d is a rotation matrix
  - scale: $A^d = cI$
  - other: A is a general matrix
- ◇ Example:



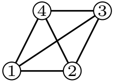
Nominal



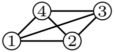
Rotation

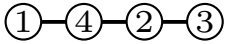


Scaling



Shear





- ◇ How to generate $A^d(t)$ and $b^d(t)$
- ◇ Affine transformation: very general

Stress Matrix

⋄ Stress ω: a stress is a set of scalars, $\{\omega_{ij}\}_{(i,j)\in\mathcal{E}}$ where $\omega_{ij} = \omega_{ji} \in \mathbb{R}$, assigned to all the edges.

⋄ Equilibrium stress:

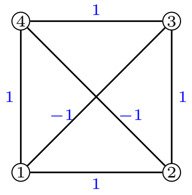
$$\sum_{j\in\mathcal{N}_i} \omega_{ij}(p_j - p_i) = 0, \qquad i \in \mathcal{V}.$$

⋄ Physical meaning:

- $\omega_{ij} > 0$ represents an attracting force
- $\omega_{ij} < 0$ represents a repelling force

The vector $\omega_{ij}(x_j - x_i)$ represents the force applied on agent i by agent j through edge (i,j). The forces applied on joint i by joints $j \in \mathcal{N}_i$ are balanced.

⋄ Example:



◇ Equilibrium stress:

$$\sum_{j\in\mathcal{N}_i} \omega_{ij}(p_j - p_i) = 0, \qquad i \in \mathcal{V}.$$

◇ Matrix form as

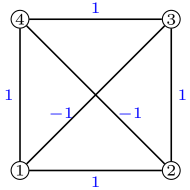
$$(\Omega \otimes I_d)p = 0,$$

where $\Omega \in \mathbb{R}^{n\times n}$ is called the *stress matrix* satisfying

$$[\Omega]_{ij} = \begin{cases} 0, & i \neq j,\ (i,j) \notin \mathcal{E}, \\ -\omega_{ij}, & i \neq j,\ (i,j) \in \mathcal{E}, \\ \sum_{k\in\mathcal{N}_i} \omega_{ik}, & i = j. \end{cases}$$

◇ Stress matrix is a generalized graph Laplacian where the weight for an edge can be positive, negative, or zero.

◇ Example:



$$\Omega = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Stress Matrix

Assumption (Stress Matrix of the Nominal Formation)

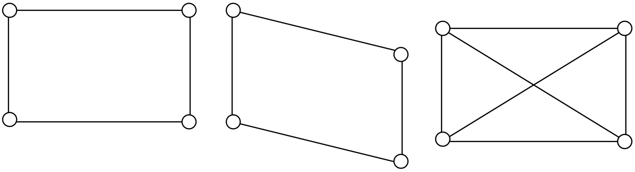
For the nominal formation $(\mathcal{G}, p^d)$, assume there exists a positive semi-definite stress matrix Ω satisfying $\mathrm{rank}(\Omega) = n - d - 1$.

- ◇ Explain later: why this assumption
- ◇ Conditions that satisfy the assumption:

Lemma (Sufficient condition)

Given an undirected graph $\mathcal{G}$ and a **generic configuration** *p, then the formation $(\mathcal{G}, p)$ is* **universally rigid** *if and only if there exists a stress matrix Ω such that Ω is positive semi-definite and $\mathrm{rank}(\Omega) = d - n - 1$.*

Details on rigidity theory are omitted. Example:



Stress matrix

Lemma (Properties of the Nominal Formation)

Under Assumption 1, if $\{p_i^d\}_{i=1}^n$ affinely span $\mathbb{R}^d$, the stress matrix of the nominal formation satisfies

$$\mathrm{Null}(\Omega \otimes I_d) = \mathcal{A}(p^d)$$

where

$$\begin{aligned}\mathcal{A}(p^d) &= \left\{p' \in \mathbb{R}^{dn} : p'_i = Ap_i^d + b, i \in \mathcal{V}, \forall A \in \mathbb{R}^{d\times d}, \forall b \in \mathbb{R}^d\right\} \\ &= \left\{p' \in \mathbb{R}^{dn} : p' = (I_n \otimes A)p^d + \mathbf{1}_n \otimes b, \forall A \in \mathbb{R}^{d\times d}, \forall b \in \mathbb{R}^d\right\}\end{aligned}$$

◇ Why this is important:

$$p^*(t) \in \mathcal{A}(p^d)$$

where $p^*(t) = [I_n \otimes A^d(t)]p^d + \mathbf{1}_n \otimes b^d(t)$ is the target formation.

The simplest affine formation control law

◇ The simplest affine formation control law [Lin et al., 2016]:

$$\dot{p}_i = -\sum_{j \in \mathcal{N}_i} \omega_{ij}(p_i - p_j), \quad i \in \mathcal{V}.$$

◇ The matrix-vector form is

$$\dot{p} = -(\Omega \otimes I_d)p.$$

◇ Key properties of Ω under the assumption:

- Stability of Ω: positive semi-definite
- Null space of Ω: $\text{Null}(\Omega \otimes I_d) = \mathcal{A}(p^d)$

◇ Convergence result:

- $p(t)$ converges to a point in $\mathcal{A}(p^d)$

◇ Problem to solve:

- converge to desired trajectories $p^*(t) \in \mathcal{A}(p^d)$.
- Solution: maneuvering leaders

Leader selection: how many and which agents should be leaders?

Definition (Affine Localizability)

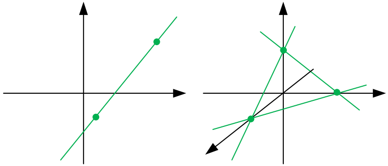
The formation $(\mathcal{G}, p^d)$ is *affinely localizable* if for any $p = [p_\ell^T, p_f^T]^T \in \mathcal{A}(p^d)$, the value of p_f can be uniquely determined by p_ℓ.

Theorem (Necessary and Sufficient Condition 1)

Under Assumption 1, formation $(\mathcal{G}, p^d)$ is affinely localizable if and only if $n_\ell \geq d+1$ and $\{p_i^d\}_{i=1}^{n_\ell}$ affinely span $\mathbb{R}^d$.

◇ Affine span

- Definition: The *affine span* of $\{p_i\}_{i=1}^n$ is the set of point $\sum_{i=1}^n a_i p_i$ for all $\{a_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n a_i = 1$.
- Geometric meaning:



Leader selection: how many and which agents should be leaders?

Let $\bar{\Omega} = \Omega \otimes I_d$. Partition $\bar{\Omega}$ according to the partition of leaders and followers as

$$\bar{\Omega} = \begin{bmatrix} \bar{\Omega}_{\ell\ell} & \bar{\Omega}_{\ell f} \\ \bar{\Omega}_{f\ell} & \bar{\Omega}_{ff} \end{bmatrix},$$

where $\bar{\Omega}_{ff} \in \mathbb{R}^{(dn_f)\times(dn_f)}$.

Theorem (Necessary and Sufficient Condition 2)

Under Assumption 1, formation $(\mathcal{G}, p^d)$ is affinely localizable if and only if Ω_{ff} is nonsingular. When Ω_{ff} is nonsingular, p_f can be determined as $p_f = -\bar{\Omega}_{ff}^{-1}\bar{\Omega}_{f\ell}p_\ell$.

Revisit the affine formation control law

Leader-follower control law:

$$\dot{p}_i = -\sum_{j\in\mathcal{N}_i}\omega_{ij}(p_i - p_j), \quad i \in \mathcal{V}_f.$$

The matrix-vector form is

$$\dot{p}_f = -\bar{\Omega}_{ff}p_f - \bar{\Omega}_{f\ell}p_\ell^*.$$

Theorem (Convergence Result)

Under Assumptions 1, if the leader velocity $\dot{p}_\ell^*(t)$ is constantly zero, then the tracking error $\delta_{p_f}(t)$ converges to zero globally and exponentially fast.

Proof.

Tracking error: $\delta_{p_f}(t) = p_f(t) - p_f^*$.

$$\dot{\delta}_{p_f} = \dot{p}_f(t) + \bar{\Omega}_{f\ell}\dot{p}_\ell^* = -\bar{\Omega}_{ff}\delta_{p_f} + \bar{\Omega}_{f\ell}\dot{p}_\ell^*.$$

Since $\dot{p}_\ell^* = 0$, the tracking error δ_{p_f} is globally and exponentially stable if and only if $\bar{\Omega}_{ff}$ is nonsingular. □

Other control laws

- Proportional-integral control:

$$\dot{p}_i = \underbrace{-\alpha \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i - p_j)}_{\text{proportional term}} \underbrace{- \beta \int_0^t \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i(\tau) - p_j(\tau))\mathrm{d}\tau}_{\text{integral term}}, \quad i \in \mathcal{V}_f,$$

- Double integrator model: position and velocity feedback

$$\begin{aligned} \dot{p}_i &= v_i, \\ \dot{v}_i &= - \sum_{j\in\mathcal{N}_i} \omega_{ij} \left[k_p(p_i - p_j) + k_v(v_i - v_j) \right], \; i \in \mathcal{V}_f, \end{aligned}$$

- Double integrator model: position, velocity, and acceleration feedback

$$\begin{aligned} \dot{p}_i &= v_i, \\ \dot{v}_i &= -\frac{1}{\gamma_i} \sum_{j\in\mathcal{N}_i} \omega_{ij} \left[k_p(p_i - p_j) + k_v(v_i - v_j) - \dot{v}_j \right], \end{aligned}$$

where $\gamma_i = \sum_{j\in\mathcal{N}_i} \omega_{ij}$.

Other control laws

◇ Unicycle model:

$$\begin{aligned} \dot{x}_i &= v_i \cos\theta_i, \\ \dot{y}_i &= v_i \sin\theta_i, \\ \dot{\theta}_i &= w_i, \end{aligned} \tag{1}$$

subject to linear and angular velocity saturation constraints:

$$\begin{aligned} -\mathrm{v}_i^{\mathrm{b}} \leq & v_i \leq \mathrm{v}_i^{\mathrm{f}}, \\ -\mathrm{w}_i^{\mathrm{r}} \leq & w_i \leq \mathrm{w}_i^{\mathrm{l}}, \end{aligned}$$

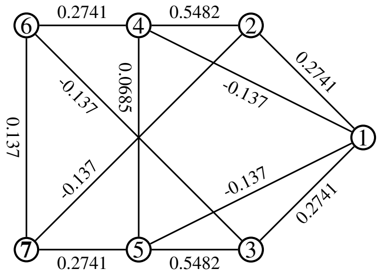
◇ Control law:

$$v_i = \mathrm{sat}_{v_i}\left\{[\cos\theta_i, \sin\theta_i] \sum_{j \in \mathcal{N}_i} \omega_{ij}(p_j - p_i)\right\},$$

$$w_i = \mathrm{sat}_{w_i}\left\{[-\sin\theta_i, \cos\theta_i] \sum_{j \in \mathcal{N}_i} \omega_{ij}(p_j - p_i)\right\}.$$

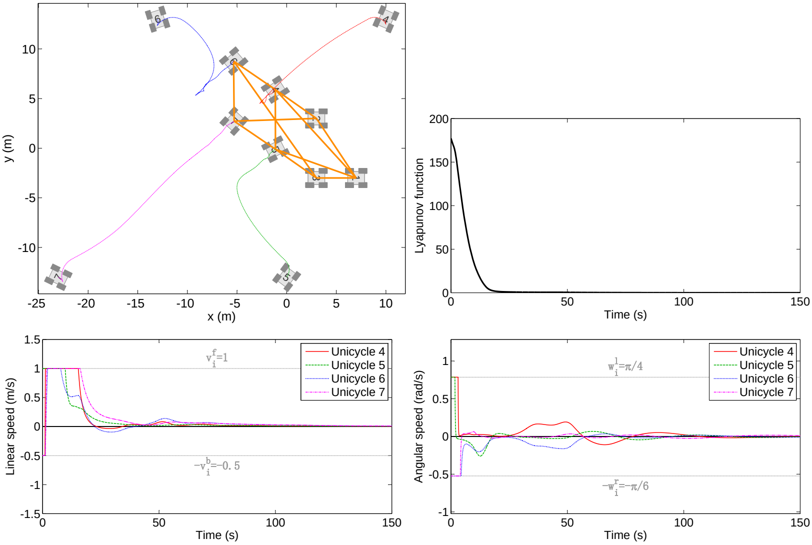
Simulation examples

Nominal formation:



The stress matrix is positive semi-definite and the eigenvalues are $\{1.4432, 1.3218, 0.5967, 0.3383, 0, 0, 0\}$.
play a video for the double-integrator agents

Simulation examples



Conclusion

This talk introduced some leader-follower affine formation maneuver control laws:

- achieve affine transformations
- can be implemented in local reference frames
- applicable to formation control in arbitrary dimensions

Main contribution:

- Leader selection
- Various linear/nonlinear control laws

Stress matrix:

- Generalized Laplacian matrix (negative edge weights)
- Positive definiteness and null space

Future work:

- Directed case

Z. Lin, L. Wang, Z. Chen, M. Fu, and Z. Han. Necessary and sufficient graphical conditions for affine formation control. *IEEE Transactions on Automatic Control*, 61(10):2877–2891, October 2016.